Preprint typeset in JHEP style - HYPER VERSION

Braiding and fusion properties of the Neveu-Schwarz super-conformal blocks

Damian Chorążkiewicz*

Institute of Theoretical Physics, University of Wrocław, pl. M. Borna 1, 95-204 Wrocław, Poland

Leszek Hadasz†

M. Smoluchowski Institute of Physics, Jagiellonian University, W. Reymonta 4, 30-059 Kraków, Poland

Abstract: We construct, generalizing appropriately the method applied by J. Teschner in the case of the Virasoro conformal blocks, the braiding and fusion matrices of the Neveu-Schwarz super-conformal blocks. Their properties allow for an explicit verification of the bootstrap equation in the NS sector of the $N = 1$ supersymmetric Liouville field theory.

Keywords: N=1 NS algebra, chiral vertex operator, conformal blocks, supersymmetric Liouville field theory.

*e-mail: damian.ch@ift.uni.wroc.pl

†e-mail: hadasz@th.if.uj.edu.pl; corresponding author

Contents

1. Introduction

In the BPZ approach to the two dimensional Conformal Field Theory [1] the rôle of a dynamical principle is played by the associativity of the operator algebra involved or, equivalently, by the bootstrap (crossing symmetry) equation imposed on the correlation functions. In every specific CFT with known spectrum and three point coupling constants, validity of the bootstrap equation can be regarded as the basic consistency condition.

Analysis of the bootstrap equation becomes particulary interesting (even if difficult) for the CFT-s with continuous spectrum. One of a few interacting and solvable models of this kind is the Liouville Field Theory. Its three point coupling constants have been found independently by Dorn and Otto [2] and by A. Zamolodchikov and Al. Zamolodchikov [3]. The authors of [3] also performed some numerical checks of the bootstrap equation in the

LFT using a recursive representation of conformal blocks developed in a series of papers by Al. Zamolodchikov [4–6].

An analytic proof of this equation which combined a Moore-Seiberg formalism of CFT [7] with a representation theory of quantum groups has been presented in [8, 9]. Using the results on fusion of degenerate representation of the Virasoro algebra the authors of [8, 9] derived from the consistency conditions of the Moore-Seiberg type a set of functional equations for the fusion matrix of the conformal blocks. These equations were then shown to be satisfied by the Racah-Wiegner coefficients for an appropriate continuous series of representations of $\mathrm{U}_q(\mathrm{sl}(2,\mathbb{R}))$. Another proof of the validity of bootstrap equation for the Liouville field theory, which relies on an explicit calculation of the fusion matrix for the conformal blocks appearing in the LFT by relating it to the braiding matrix of Virasoro chiral vertex operators, was presented in [10, 11].

Conformal field theory with $N = 1$ supersymmetry [12–14] is in some sense the simplest generalization of the "ordinary" CFT. Also here the bulk three point coupling constants of the basic interacting model — supersymmetric extension of the Liouville theory — are known [15] and some numerical checks of the bootstrap equation in the Neveu-Schwarz sector of the theory (which employed a recursive representation of $N = 1$ NS blocks developed in [16, 17]) have been performed [18, 19]. However, an analytical proof of the consistency of the $N = 1$ supersymmetric Liouville theory is still missing.

A step towards such a proof was taken in [20], where the form of the fusion matrix for the Neveu-Schwarz superconformal blocks was postulated. If correct, it implies the bootstrap equation for four super-primary NS Liouville fields. The basic goal of the preset work is to justify the results of [20] by calculating explicitly (in the spirit of [10, 11]) the braiding matrices of the NS chiral operators and relating them to the fusion matrices of NS superconformal blocks.

The paper is organized as follows. In the Section 2 we briefly discuss the Neveu-Schwarz algebra and recall the basic facts on the $N = 1$ supersymmetric Liouville field theory. This section is in a sense introductory and mainly meant to establish a convenient notation. In the Section 3 we discuss the structure of the highest weight NS super-moduli, define the NS chiral vertex operators and postulate for them the existence of a braiding relation. Section 4, which constitutes the main part of the present work, is devoted to an explicit construction of the braiding matrix of the NS chiral vertex operators by relating it to an exchange matrix of the screened, normal ordered exponential build up from the modes of the chiral superscalar field. In the Section 5 we introduce the NS blocks and discuss their braiding and fusion properties. Coincidence of the result with the fusion matrix "guessed" in [20] may be viewed as a proof of the validity of the bootstrap equation in the NS sector of the supersymmetric Liouville field theory. Finally, Appendix A contains some relevant properties of the Barnes double gamma function and Appendix B is devoted to a derivation of a Weyl-type representation of the screening charge operators.

2. Bootstrap in the $N = 1$ supersymmetric Liouville field theory

The $N = 1$ supersymmetric Liouville field theory (see [21] for an exhaustive review) may be defined by the action

$$\mathcal{S}_{\rm SLFT} = \int d^2z \left(\frac{1}{2\pi} \left|\partial\phi\right|^2 + \frac{1}{2\pi} \left(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}\right) + 2i\mu b^2 \bar{\psi}\psi \mathrm{e}^{b\phi} + 2\pi b^2\mu^2 \mathrm{e}^{2b\phi} \right),$$

where ϕ is a bosonic and ψ a fermionic field, μ denotes a two-dimensional cosmological constant and b is a Liouville coupling constant.

The superconformal symmetry of the SLFT (as well an of any other $N = 1$ superconformal filed theory) is generated by a pair of holomorphic currents $T(z)$, $S(z)$ and their anti-holomorphic counterparts $\overline{T}(\bar{z})$, $\overline{S}(\bar{z})$, where T and $\overline{T}$ are components of the energy-momentum tensor while S and $\overline{S}$ have dimensions $(3/2, 0)$ and $(0, 3/2)$, respectively. The algebra of the modes of $T(z)$ and $S(z)$ is determined by the operator product expansion (OPE):

$$\begin{aligned} T(z)T(0) &= \frac{c}{2z^4} + \frac{2}{z^2}T(z) + \frac{1}{z}\partial T(0) + \ldots, \\ T(z)S(0) &= \frac{3}{2z^2}S(0) + \frac{1}{z}\partial S(0) + \ldots, \\ S(z)S(0) &= \frac{2c}{3z^3} + \frac{2}{z}T(0) + \ldots . \end{aligned} \tag{2.1}$$

The fields local with respect to $S(z)$, i.e. with the OPE

$$S(z)\phi_{\rm NS}(0,0) \;=\; \sum_{k\in\mathbb{Z}+\frac{1}{2}} z^{k-\frac{3}{2}} S_{-k}\phi_{\rm NS}(0,0)$$

form the Neveu-Schwarz (or NS for brevity) subspace in a space of fields of a given SCFT on which we shall focus in the present paper. Together with the usual Virasoro generators L_n defined by the OPE

$$T(z)\phi_{\rm NS}(0,0) \;=\; \sum_{n\in\mathbb{Z}} z^{n-2} L_{-n}\phi_{\rm NS}(0,0),$$

S_k form the Neveu-Schwarz algebra determined by (2.1),

$$\begin{aligned} \left[L_m, L_n\right] &= (m-n)L_{m+n} + \frac{c}{12}m\left(m^2-1\right)\delta_{m+n}, \\ \left[L_m, S_k\right] &= \frac{m-2k}{2}S_{m+k}, \\ \{S_k, S_l\} &= 2L_{k+l} + \frac{c}{3}\left(k^2 - \frac{1}{4}\right)\delta_{k+l}. \end{aligned} \tag{2.2}$$

It is convenient to parameterize the central charge c of the NS algebra as

$$c = \frac{3}{2} + 3Q^2,$$

where in the case of SLFT the "background charge" Q my be expressed through the Liouville coupling constant as

$$Q = b + b^{-1}.$$

In the space of NS fields there exist "super-primary" fields, realized in the supersymmetric Liouville theory as an appropriately normal ordered exponents $V_a(z,\bar{z}) = \mathrm{e}^{a\varphi(z,\bar{z})}$. By definition they satisfy:

$$\begin{aligned} [L_n, V_a(0,0)] \; &= \; [S_k, V_a(0,0)] \; = \; 0, \qquad\quad n,k>0, \\ [L_0, V_a(0,0)] \; &= \; \Delta_a V_a(0,0), \qquad \Delta_a = \frac{1}{2}a(Q-a). \end{aligned}$$

Each super-primary field is the "lowest" component of the superfield

$$\Phi_a(z,\theta;\bar{z},\bar{\theta}) \;=\; V_a(z,\bar{z}) + \theta\,\Lambda_a(z,\bar{z}) + \bar{\theta}\,\bar{\Lambda}_a(z,\bar{z}) - \theta\bar{\theta}\,\widetilde{V}_a(z,\bar{z}), \tag{2.3}$$

where

$$\Lambda_a \;=\; \left[S_{-1/2}, V_a\right], \qquad \bar{\Lambda}_a \;=\; \left[\bar{S}_{-1/2}, V_a\right], \qquad \widetilde{V}_a \;=\; \left\{S_{-1/2}, \left[\bar{S}_{-1/2}, V_a\right]\right\},$$

and $\theta, \bar{\theta}$ are Grassman numbers. Global superconformal transformations (generated by $L_0, S_{\pm\frac{1}{2}}, L_{\pm 1}$ and their right counterparts) allow to express three-point function of primary superfields in the form:

$$\begin{aligned} &\Big\langle \Phi_{a_3}(z_3,\theta_3;\bar{z}_3,\bar{\theta}_3)\Phi_{a_2}(z_2,\theta_2;\bar{z}_2,\bar{\theta}_2)\Phi_{a_1}(z_1,\theta_1;\bar{z}_1,\bar{\theta}_1)\Big\rangle \\ &\qquad = \; Z_{32}^{\gamma_1}\bar{Z}_{32}^{\bar{\gamma}_1}\,Z_{31}^{\gamma_2}\bar{Z}_{31}^{\bar{\gamma}_2}\,Z_{21}^{\gamma_3}\bar{Z}_{21}^{\bar{\gamma}_3}\,\Big\langle \Phi_{a_3}(\infty,0;\infty,0)\Phi_{a_2}(1,\Theta;1,\bar{\Theta})\Phi_{a_1}(0,0;0,0)\Big\rangle, \end{aligned}$$

where $\gamma_i = 2\Delta_i - (\Delta_1+\Delta_2+\Delta_3)$, $\;Z_{ij} = z_i - z_j - \theta_i\theta_j \equiv z_{ij} - \theta_i\theta_j$,

$$\Theta \;=\; \frac{1}{\sqrt{z_{12}z_{13}z_{23}}}\left(\theta_1 z_{23} + \theta_2 z_{31} + \theta_3 z_{12} - \frac{1}{2}\theta_1\theta_2\theta_3\right),$$

is an odd invariant of the global superconformal group and

$$\Phi_{a_3}(\infty,0;\infty,0) \equiv \lim_{R\to\infty} R^{2\Delta_3+2\bar{\Delta}_3}\Phi_{a_3}(R,0;R,0).$$

The three point function is thus determined by the superconformal symmetry up to two independent constants,

$$\begin{aligned} C(a_3,a_2,a_1) \;&=\; \big\langle V_{a_3}(\infty,\infty)V_{a_2}(1,1)V_{a_1}(0,0)\big\rangle, \\ \widetilde{C}(a_3,a_2,a_1) \;&=\; \big\langle V_{a_3}(\infty,\infty)\widetilde{V}_{a_2}(1,1)V_{a_1}(0,0)\big\rangle. \end{aligned}$$

Their form in the $N=1$ supersymmetric Liouville field theory,

$$C(a_3,a_2,a_1) \;=\; C_0(a)\,\frac{\Upsilon_{\mathrm{NS}}(2a_3)\Upsilon_{\mathrm{NS}}(2a_2)\Upsilon_{\mathrm{NS}}(2a_1)}{\Upsilon_{\mathrm{NS}}(a-Q)\Upsilon_{\mathrm{NS}}(a_{1+2-3})\Upsilon_{\mathrm{NS}}(a_{2+3-1})\Upsilon_{\mathrm{NS}}(a_{3+1-2})},$$

$$\widetilde{C}(a_3,a_2,a_1) \;=\; 2i\,C_0(a)\,\frac{\Upsilon_{\mathrm{NS}}(2a_3)\Upsilon_{\mathrm{NS}}(2a_2)\Upsilon_{\mathrm{NS}}(2a_1)}{\Upsilon_{\mathrm{R}}(a-Q)\Upsilon_{\mathrm{R}}(a_{1+2-3})\Upsilon_{\mathrm{R}}(a_{2+3-1})\Upsilon_{\mathrm{R}}(a_{3+1-2})},$$

with

$$C_0(a) = \left(\pi\mu\gamma\left(\frac{bQ}{2}\right)b^{1-b^2}\right)^{\frac{Q-a}{b}}\Upsilon'_{\rm NS}(0),$$

was first derived in [15]. Here $a \equiv a_1 + a_2 + a_3$, $a_{1+2-3} \equiv a_1 + a_2 - a_3$, etc. and the special functions involved ($\Upsilon_{\rm NS,R}(x), G_{\rm NS,R}(x)$ below etc.) are defined in Appendix A.

The super-projective transformations also allow to express a generic function of four superfields (2.3) through the four-point functions of the form

$$\Big\langle \Phi_{a_4}(\infty,0;\infty,0)\Phi_{a_3}(1,\theta_3;1,\bar\theta_3)\Phi_{a_2}(z,\theta_2;\bar z,\bar\theta_2)\Phi_{a_1}(0,0;0,0)\Big\rangle, \tag{2.4}$$

If we denote by $\mathcal{F}^{\rm e}_{a_s}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right](z)$ and $\mathcal{F}^{\rm o}_{a_s}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right](z)$ a pair (out of four) of an even and an odd $N = 1$ Neveu-Schwarz blocks[1] then a special case of (2.4), the four point function of super-primary fields

$$G_4(z,\bar z) = \big\langle V_{a_4}(\infty,\infty)V_{a_3}(1,1)V_{a_2}(z,\bar z)V_{a_1}(0,0)\big\rangle,$$

can be presented either in the "$s-$channel":

$$\begin{aligned} G_4(z,\bar z) \;=\; \int\limits_{\mathbb{S}}\frac{da_s}{i}\,\Big[&C(a_4,a_3,a_s)C(\bar a_s,a_2,a_1)\left|\mathcal{F}^{\rm e}_{a_s}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right](z)\right|^2 \\ &-\widetilde{C}(a_4,a_3,a_s)\widetilde{C}(\bar a_s,a_2,a_1)\left|\mathcal{F}^{\rm o}_{a_s}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right](z)\right|^2\Big]\end{aligned} \tag{2.5}$$

with $\mathbb{S} = \frac{Q}{2} + i\mathbb{R}_+$, or in the "$t-$channel" decomposition:

$$\begin{aligned} G_4(z,\bar z) \;=\; \int\limits_{\mathbb{S}}\frac{da_t}{i}\,\Big[&C(a_4,a_t,a_1)C(\bar a_t,a_2,a_3)\left|\mathcal{F}^{\rm e}_{a_t}\left[\begin{smallmatrix} a_1 & a_2\\ a_4 & a_3\end{smallmatrix}\right](1-z)\right|^2 \\ &-\widetilde{C}(a_4,a_t,a_1)\widetilde{C}(\bar a_t,a_2,a_3)\left|\mathcal{F}^{\rm o}_{a_t}\left[\begin{smallmatrix} a_1 & a_2\\ a_4 & a_3\end{smallmatrix}\right](1-z)\right|^2\Big].\end{aligned} \tag{2.6}$$

Here and in what follows we use a convenient notation $\bar a = Q - a$ (notice that for $a \in \frac{Q}{2} + i\mathbb{R}$ it is indeed the complex conjugate of a). If we now define a fusion matrix F by assuming the existence of a relation between the blocks appearing in the decompositions above,

$$\mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right](z) = \int\limits_{\mathbb{S}}\frac{da_t}{2i}\sum_{\rho={\rm e,o}}\mathsf{F}_{a_s a_t}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\eta}{}_{\rho}\,\mathcal{F}^{\rho}_{a_t}\left[\begin{smallmatrix} a_1 & a_2\\ a_4 & a_3\end{smallmatrix}\right](1-z), \qquad \eta = {\rm e,o},$$

then the coincidence of the decompositions (2.5) and (2.6) may be recast in the form

$$\begin{aligned}\int\limits_{\mathbb{S}}\frac{da_s}{i}\;&\left(\mathsf{F}_{a_s a_t}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]\right)^{\dagger}\cdot\mathsf{C}(a_4,a_3,a_s)\cdot\tau_3\cdot\mathsf{C}(\bar a_s,a_2,a_1)\cdot\mathsf{F}_{a_s a'_t}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right] \\ &=\;\mathsf{C}(a_4,a_t,a_1)\cdot\tau_3\cdot\mathsf{C}(\bar a_t,a_3,a_2)\;i\delta(a_t-a'_t).\end{aligned} \tag{2.7}$$

where we denoted

$$\mathsf{C}(a_3,a_2,a_1) \;=\; \begin{pmatrix} C(a_3,a_2,a_1) & 0\\ 0 & \widetilde{C}(a_s,a_2,a_1)\end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

[1]See [16, 18] or Section 5 for the definitions.

Coincidence of the $s-$ and $t-$channel representations of the four-point correlation function, or (equivalently) Eq. (2.7), constitutes the bootstrap equation for the super-primary fields. Analogous procedure can be applied also to the other four-point correlation functions appearing in (2.4), what results in the remaining bootstrap equations for NS sector of the supersymmetric Liouville field theory.

3. Chiral vertex operators

3.1 Neveu-Schwarz super-module

Let $\varphi_a(0,0)$ denotes a NS superprimary field. States

$$\nu_a = \varphi_a(0,0)|0\rangle, \tag{3.1}$$

obtained through its action on the invariant vacuum $|0\rangle$, with

$$L_n|0\rangle = S_k|0\rangle = 0, \qquad n \geqslant -1, \quad k \geqslant -\tfrac{1}{2},$$

are of the highest weight with respect to the NS algebra (2.2):

$$L_0\nu_a = \Delta_a\nu_a\,, \qquad L_n\nu_a = S_k\nu_a = 0, \qquad n,k > 0. \tag{3.2}$$

We shall frequently write ν_0 instead of $|0\rangle$; this is consistent with (3.1) since the unique super-primary field with the conformal weight 0 is the identity operator, $\varphi_0(0,0) = \mathbf{1}$.

Denote by $\mathcal{V}_a^f$ the free vector space generated by all vectors of the form

$$\nu_{a,NK} \;=\; L_{-N}S_{-K}\nu_a \;\equiv\; L_{-n_j}\ldots L_{-n_1}S_{-k_i}\ldots S_{-k_1}\nu_a\,, \tag{3.3}$$

where $K = \{k_1, k_2, \ldots, k_i\}$ and $N = \{n_1, n_2, \ldots, n_j\}$ are arbitrary ordered sets of indices

$$k_i > \ldots > k_2 > k_1, \qquad n_j \geqslant \ldots \geqslant n_2 \geqslant n_1,$$

such that $|K| + |N| \equiv k_1 + \cdots + k_i + n_1 + \cdots + n_j = f$.

The $\frac{1}{2}\mathbb{Z}$-graded representation of the NS algebra, determined on the space

$$\mathcal{V}_a \;=\; \bigoplus_{f\in\frac{1}{2}\mathbb{N}} \mathcal{V}_a^f\,, \qquad \mathcal{V}_a^0 = \mathbb{C}\,\nu_a\,,$$

by the relations (2.2) and (3.2), is called the NS supermodule of the highest weight Δ_a and the central charge c (to avoid making the notation overloaded we omit the subscript c at $\mathcal{V}$). Each $\mathcal{V}_a^f$ is an eigenspace of L_0 with the eigenvalue $\Delta_a + f$. The space $\mathcal{V}_a$ has also a natural $\mathbb{Z}_2$-grading:

$$\mathcal{V}_a \;=\; \mathcal{V}_a^+ \oplus \mathcal{V}_a^-\,, \qquad \mathcal{V}_a^+ \;= \bigoplus_{n\in\mathbb{N}} \mathcal{V}_a^n\,, \qquad \mathcal{V}_a^- \;= \bigoplus_{k\in\mathbb{N}+\frac{1}{2}} \mathcal{V}_a^k\,,$$

where $\mathcal{V}_a^\pm$ are eigenspaces of the parity operator $(-1)^{\mathsf{F}} = (-1)^{2(L_0-\Delta_a)}$. Finally, there exists on $\mathcal{V}_a$ a natural, symmetric, bilinear form

$$\langle\,\cdot\,|\,\cdot\,\rangle_a \;:\; \mathcal{V}_a \times \mathcal{V}_a \;\to\; \mathbb{C},$$

uniquely determined by the algebra (2.2), normalization $\langle \nu_a|\nu_a\rangle_a = 1$ and the relations $(L_n)^\dagger = L_{-n},\ (S_k)^\dagger = S_{-k}$. In what follows we will usually suppress the index a at $\langle\,\cdot\,|\,\cdot\,\rangle_a$.

3.2 The vertex

The NS chiral vertex operator ${}^{\mathbf{N}}V_{\mathbb{A}}(z)$, where $\mathbb{A} = \left({}_{a_3}{}^{a_2}{}_{a_1}\right)$ and $z \in \mathbb{C}$, is a linear map from $\mathcal{V}_{a_1} \equiv \mathcal{V}_1$ to $\mathcal{V}_3$ which may be defined by the following conditions[2]:

1. ${}^{\mathbf{N}}V_{\mathbb{A}}(z)$ is a sum of an even (i.e. parity preserving) and an odd (i.e. parity reversing) operators,
$$ {}^{\mathbf{N}}V_{\mathbb{A}}(z) = {}^{\mathbf{N}}V^{\mathrm{e}}_{\mathbb{A}}(z) + {}^{\mathbf{N}}V^{\mathrm{o}}_{\mathbb{A}}(z),$$
where ${}^{\mathbf{N}}V^{\mathrm{e}}_{\mathbb{A}}(z) \;:\; \mathcal{V}_1^\pm \;\to\; \mathcal{V}_3^\pm$ and ${}^{\mathbf{N}}V^{\mathrm{o}}_{\mathbb{A}}(z) \;:\; \mathcal{V}_1^\pm \;\to\; \mathcal{V}_3^\mp$.

2. Let θ be an anticommuting variable,
$$\{\theta,\theta\} = \{\theta, S_k\} = \{\theta, {}^{\mathbf{N}}V^{\mathrm{o}}_{\mathbb{A}}(z)\} = [\theta, L_n] = \left[\theta, {}^{\mathbf{N}}V^{\mathrm{e}}_{\mathbb{A}}(z)\right] = 0.$$
Then
$$\begin{aligned} \left[L_n, {}^{\mathbf{N}}V_{\mathbb{A}}(z)\right] &= z^n\left(z\partial_z + (n+1)\Delta_2\right) {}^{\mathbf{N}}V_{\mathbb{A}}(z), \\ \left[\theta S_k, {}^{\mathbf{N}}V_{\mathbb{A}}(z)\right] &= z^{k+\frac{1}{2}}\left[\theta S_{-1/2}, {}^{\mathbf{N}}V_{\mathbb{A}}(z)\right]. \end{aligned} \tag{3.4}$$

3. The commutation relations (3.4) determine ${}^{\mathbf{N}}V_{\mathbb{A}}(z)$ up to two arbitrary functions of the parameters a_1, a_2, a_3. For $z \to 0$:
$$ {}^{\mathbf{N}}V_{\mathbb{A}}(z)\nu_1 \;=\; z^{\Delta_3-\Delta_2-\Delta_1}\left(N^{\mathrm{e}}(a_3,a_2,a_1)\nu_3 + z^{\frac{1}{2}}\frac{N^{\mathrm{o}}(a_3,a_2,a_1)}{2\Delta_3}S_{-\frac{1}{2}}\nu_3 + \mathcal{O}(z)\right), \tag{3.5}$$
so that
$$N^{\mathrm{e}}(a_3,a_2,a_1) \;=\; \left\langle \nu_3\middle|{}^{\mathbf{N}}V_{\mathbb{A}}(z)\nu_1\right\rangle, \quad N^{\mathrm{o}}(a_3,a_2,a_1) \;=\; \left\langle S_{-\frac{1}{2}}\nu_3\middle|{}^{\mathbf{N}}V_{\mathbb{A}}(z)\nu_1\right\rangle.$$

We shall define a *normalized* NS chiral vertex operator $V_{\mathbb{A}}(z)$ to be the NS chiral vertex operator with $N^{\mathrm{e}}(a_3,a_2,a_1) = N^{\mathrm{o}}(a_3,a_2,a_1) = 1$. Equivalently, for any ${}^{\mathbf{N}}V_{\mathbb{A}}(z)$:

$$ {}^{\mathbf{N}}V^{\rho}_{\mathbb{A}}(z) \;=\; N^{\rho}(a_3,a_2,a_1)\,V^{\rho}_{\mathbb{A}}(z), \qquad \rho = \mathrm{e},\ \mathrm{o}. \tag{3.6}$$

[2] The basic facts on the vertex operators can be found in [7]; [22] and [23] can be consulted for a clear and extensive introduction to the subject. The presented formulation parallels the one used in [10].

The operator ${}^{\mathbf{N}}V_{\mathbb{A}}(z)$ is naturally associated with the ground state $\nu_2 \in \mathcal{V}_2$. It is useful to define a family of (generalized) NS chiral vertex operators enumerated by (and linear in) vectors $\xi \in \mathcal{V}_2$. If we denote ${}^{\mathbf{N}}V_{a_3a_1}(\nu_2|z) = {}^{\mathbf{N}}V_{\mathbb{A}}(z)$ then, first of all,

$$
\begin{aligned}
\theta\, {}^{\mathbf{N}}V_{a_3a_1}(S_{-1/2}\nu_2|z) &= \left[\theta S_{-1/2}, {}^{\mathbf{N}}V_{a_3a_1}(\nu_2|z)\right], \\
{}^{\mathbf{N}}V_{a_3a_1}(L_{-1}\xi|z) &= \frac{\partial}{\partial z}\, {}^{\mathbf{N}}V_{a_3a_1}(\xi|z), \qquad \xi \in \mathcal{V}_2.
\end{aligned}
\tag{3.7}
$$

Moreover, for $k \in \mathbb{Z} + \frac{1}{2},\ k \geqslant 3/2$:

$$
\theta\, {}^{\mathbf{N}}V_{a_3a_1}(S_{-k}\xi|z) = \frac{1}{(k-\frac{3}{2})!}\left(\frac{\partial}{\partial w}\right)^{k-\frac{3}{2}} :\theta S(w)\, {}^{\mathbf{N}}V_{a_3a_1}(\xi|z): \Big|_{w=z}
$$

where

$$
:\theta S(w)\, {}^{\mathbf{N}}V_{a_3a_1}(\xi|z): = \Big(\sum_{l \leqslant -\frac{3}{2}} \theta S_l\, w^{-l-\frac{3}{2}}\Big) {}^{\mathbf{N}}V_{a_3a_1}(\xi|z) + {}^{\mathbf{N}}V_{a_3a_1}(\xi|z) \Big(\sum_{l \geqslant -\frac{1}{2}} \theta S_l\, w^{-l-\frac{3}{2}}\Big),
$$

with $l \in \mathbb{Z} + \frac{1}{2}$ and, for $m \geqslant 2$:

$$
{}^{\mathbf{N}}V_{a_3a_1}(L_{-m}\xi|z) = \frac{1}{(m-2)!}\left(\frac{\partial}{\partial w}\right)^{m-2} :T(w)\, {}^{\mathbf{N}}V_{a_3a_1}(\xi|z): \Big|_{w=z}
$$

with

$$
:T(w)\, {}^{\mathbf{N}}V_{a_3a_1}(\xi|z): = \Big(\sum_{n \leqslant -2} L_n\, w^{-n-2}\Big) {}^{\mathbf{N}}V_{a_3a_1}(\xi|z) + {}^{\mathbf{N}}V_{a_3a_1}(\xi|z) \Big(\sum_{n \geqslant -1} L_n\, w^{-n-2}\Big).
$$

The state-operator correspondence above is build in a way which ensures that also the operator-state correspondence,

$$
\forall\, \xi \in \mathcal{V}_a : \quad \lim_{z \to 0} V_{a0}(\xi|z)\nu_0 = \xi,
$$

holds.

Since we shall frequently use the generalized chiral vertex operator associated with the vector $S_{-1/2}\nu_2$ it is convenient to reserve for it a special notation and write

$$
{}^{\mathbf{N}}V_{a_3a_1}(S_{-1/2}\nu_2|z) \equiv {}^{\mathbf{N}}V_{a_3a_1}(*\nu_2|z).
$$

It then follows from (3.4) that

$$
\begin{aligned}
\left[\mathrm{L}_m, {}^{\mathbf{N}}V_{a_3a_1}(\nu_2|z)\right] &= z^m \left(z\partial_z + (m+1)\Delta_2\right) {}^{\mathbf{N}}V_{a_3a_1}(\nu_2|z), \\
\left[\mathrm{L}_m, {}^{\mathbf{N}}V_{a_3a_1}(*\nu_2|z)\right] &= z^m \left(z\partial_z + (m+1)\left(\Delta_2 + \tfrac{1}{2}\right)\right) {}^{\mathbf{N}}V_{a_3a_1}(*\nu_2|z), \\
\left[\mathrm{S}_k, {}^{\mathbf{N}}V^{\mathrm{e}}_{a_3a_1}(\nu_2|z)\right] &= z^{k+\frac{1}{2}}\ {}^{\mathbf{N}}V^{\mathrm{o}}_{a_3a_1}(*\nu_2|z), \\
\left\{\mathrm{S}_k, {}^{\mathbf{N}}V^{\mathrm{o}}_{a_3a_1}(\nu_2|z)\right\} &= z^{k+\frac{1}{2}}\ {}^{\mathbf{N}}V^{\mathrm{e}}_{a_3a_1}(*\nu_2|z), \\
\left[\mathrm{S}_k, {}^{\mathbf{N}}V^{\mathrm{e}}_{a_3a_1}(*\nu_2|z)\right] &= z^{k-\frac{1}{2}} \left(z\partial_z + \Delta_2(2k+1)\right)\ {}^{\mathbf{N}}V^{\mathrm{o}}_{a_3a_1}(\nu_2|z), \\
\left\{\mathrm{S}_k, {}^{\mathbf{N}}V^{\mathrm{o}}_{a_3a_1}(*\nu_2|z)\right\} &= z^{k-\frac{1}{2}} \left(z\partial_z + \Delta_2(2k+1)\right)\ {}^{\mathbf{N}}V^{\mathrm{e}}_{a_3a_1}(\nu_2|z).
\end{aligned}
\tag{3.8}
$$

Consequently

$$
\begin{aligned}
{}^{\mathbf{N}}V_{a_3a_1}(*\nu_2|z)\nu_1 \;=&\\
& z^{\Delta_3-\Delta_2-\Delta_1-\frac{1}{2}}\left(N^{\mathrm{o}}(a_3,a_2,a_1)\nu_3 + z^{\frac{1}{2}}(\Delta_3+\Delta_2-\Delta_1)N^{\mathrm{e}}(a_3,a_2,a_1)S_{-\frac{1}{2}}\nu_3 + \mathcal{O}(z)\right),
\end{aligned}
$$

so that

$$
{}^{\mathbf{N}}V^{\rho}_{a_3a_1}(*\nu_2|z) \;=\; N^{\bar{\rho}}(a_3,a_2,a_1)\,V^{\rho}_{a_3a_1}(*\nu_2|z), \tag{3.9}
$$

where $\bar{\mathrm{e}} = \mathrm{o}$, $\bar{\mathrm{o}} = \mathrm{e}$.

Let us define a braiding matrix to be an integral kernel which appears in the operator identity[3]

$$
V^{\rho}_{a_4a_s}(\xi_3|z_3)V^{\eta}_{a_sa_1}(\xi_2|z_2) = \int\limits_{\mathbb{S}} \frac{da_u}{2i} \sum_{\lambda,\delta=\,\mathrm{e,o}} \mathsf{B}^{\epsilon}_{a_sa_u}\!\left[\begin{smallmatrix}\xi_3 & \xi_2\\ a_4 & a_1\end{smallmatrix}\right]^{\rho\eta}{}_{\lambda\delta}\, V^{\lambda}_{a_4a_u}(\xi_2|z_2)V^{\delta}_{a_ua_1}(\xi_3|z_3), \tag{3.10}
$$

where $\epsilon = \mathrm{sign}(\mathrm{Arg}\, z_{32})$ with $\mathrm{sign}(x) = +1$ for $x > 0$ and -1 for $x < 0$, and in the case relevant for the supersymmetric Liouville field theory $\mathbb{S} = \frac{Q}{2} + i\mathbb{R}$. It follows from (3.8) and the definition of $V^{\rho}_{a_3a_1}(\xi|z)$ that every matrix element of $V^{\rho}_{a_4a_k}(\xi_i|z_i)V^{\eta}_{a_ka_1}(\xi_j|z_j)$ can be expressed as a linear differential operator (in z_i and z_j) acting on a matrix element of the product $V^{\rho}_{a_4a_k}(\_\nu_i|z_i)V^{\eta}_{a_ka_1}(\_\nu_j|z_j)$, where $\_\nu_l$ denotes ν_l or $*\nu_l \equiv S_{-1/2}\nu_l$. Because the braiding matrix in (3.10) does not depend on z_2 and z_3, it is then equal to one of the matrices appearing in the braiding relations for the operators $V^{\rho}_{a_4a_k}(\_\nu_i|z_i)$ and $V^{\eta}_{a_ka_1}(\_\nu_j|z_j)$:

$$
V^{\rho}_{a_4a_s}(\_\nu_3|z_3)V^{\eta}_{a_sa_1}(\_\nu_2|z_2) = \int\limits_{\mathbb{S}} \frac{da_u}{2i} \sum_{\lambda,\delta=\,\mathrm{e,o}} \mathsf{B}^{\epsilon}_{a_sa_u}\!\left[\begin{smallmatrix}\_a_3 & \_a_2\\ \_a_4 & \_a_1\end{smallmatrix}\right]^{\rho\eta}{}_{\lambda\delta}\, V^{\lambda}_{a_4a_u}(\_\nu_2|z_2)V^{\delta}_{a_ua_1}(\_\nu_3|z_3), \tag{3.11}
$$

where, in order to make the notation uniform, we have chosen as the arguments of the braiding matrix $\_a_i$ rather then $\_\nu_i$.

Graphically, we shall denote the braiding "move" as

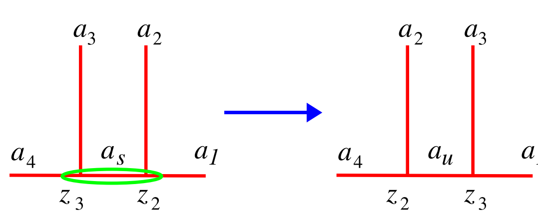

Fig. 1 Graphical notation for the braiding transformation

[3]For the rational CFT, where the integral in (3.10) is replaced by a finite sum, the existence of the braiding matrix can be proven, [7, 24]. For the $N = 1$ SLFT the assumption that a braiding matrix satisfying (3.10) does exists will be justified by it explicit construction — the primary goal of this work.

4. Chiral superscalar

It turns out to be difficult to derive an explicit form or the braiding matrix just from algebraic properties of the NS chiral vertex operators. We shall therefore construct in this section (appropriately modifying the Teschner's procedure for the Virasoro case) a rather explicit realization of the NS chiral vertex on the tensor product of a free chiral scalar and a free chiral fermion Hilbert spaces. With the properties of the braiding matrix derived we will then proceed in Section 5 to a discussion of the braiding and fusion properties of the Neveu-Schwarz conformal blocks.

4.1 Chiral fields

Let us define for $\sigma \in \mathbb{R}$ the chiral scalar field

$$\varphi(\sigma) = \mathsf{q} + \sigma \mathsf{p} + \varphi_<(\sigma) + \varphi_>(\sigma)$$

where

$$\varphi_<(\sigma) = i \sum_{n=-\infty}^{-1} \frac{\mathsf{a}_n}{n}\, \mathrm{e}^{-in\sigma}, \qquad \varphi_>(\sigma) \ = \ i \sum_{n=1}^{\infty} \frac{\mathsf{a}_n}{n}\, \mathrm{e}^{-in\sigma}.$$

The modes enjoy the usual hermitian conjugation

$$\mathsf{p}^\dagger = \mathsf{p}, \qquad \mathsf{q}^\dagger = \mathsf{q}, \qquad \mathsf{a}_n^\dagger = \mathsf{a}_{-n}, \tag{4.1}$$

as well as commutation

$$[\mathsf{q}, \mathsf{p}] = i, \qquad [\mathsf{a}_m, \mathsf{a}_n] = m\delta_{m+n},$$

properties, realized on the Hilbert space $\mathcal{H}_\mathrm{B} = L^2(\mathbb{R}) \otimes \mathcal{F}_\mathrm{B}$, where $\mathcal{F}_\mathrm{B}$ is the Fock space generated by the action of the creation operators a_{-n}, $n > 0$, on the ground state Ω_B annihilated by a_n, $n > 0$.

The second ingredient we shall need is the (antiperiodic for $\sigma \to \sigma + 2\pi$) chiral fermion field

$$\psi(\sigma) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_k\, \mathrm{e}^{-ik\sigma}, \tag{4.2}$$

with the conjugation

$$\psi_k^\dagger \ = \ \psi_{-k} \tag{4.3}$$

and anticommutation relation $\{\psi_k, \psi_l\} \ = \ \delta_{k+l}$ realized on the Fock space $\mathcal{F}_\mathrm{F}$ generated by the action of ψ_{-k} on the fermionic vacuum Ω_F, $\psi_k \Omega_\mathrm{F} = 0$, $k > 0$.

Let $\mathcal{H}_p$ be the vector space generated by the action of the creation operators a_{-n} and ψ_{-k} on the "ground state" $\tilde{\nu}_p = |p\rangle \otimes \Omega_\mathrm{B} \otimes \Omega_\mathrm{F}$, where $\mathsf{p}|p\rangle = p|p\rangle$, $p \in \mathbb{R}$. One can define

on $\mathcal{H}_p$ a standard free field representation of the Neveu-Schwarz algebra,

$$
\begin{aligned}
L_0 &= \frac{1}{8}Q^2 + \frac{1}{2}\mathsf{p}^2 + \sum_{m\geqslant 1} \mathsf{a}_{-m}\mathsf{a}_m + \sum_{k\geqslant \frac{1}{2}} k\,\psi_{-k}\psi_k, \\
L_n &= \left(\mathsf{p} + \frac{inQ}{2}\right)\mathsf{a}_n + \frac{1}{2}\sum_{m\neq 0,n} \mathsf{a}_{n-m}\mathsf{a}_m + \frac{1}{2}\sum_{k\in\mathbb{Z}+\frac{1}{2}} k\,\psi_{n-k}\psi_k, \qquad n\neq 0, \\
S_k &= (\mathsf{p} + iQk)\psi_k + \sum_{m\neq 0} \mathsf{a}_m\psi_{k-m},
\end{aligned}
$$

with the central charge $c = \frac{3}{2} + 3Q^2$ and the highest weight vector $\tilde{\nu}_p$ having the conformal weight $\frac{1}{8}Q^2 + \frac{1}{2}p^2$. If we further define on $\mathcal{H}_p$ a bilinear form $\langle\!\langle\cdot\,|\,\cdot\rangle\!\rangle_p$ such that a normalization condition $\langle\!\langle\tilde{\nu}_p|\tilde{\nu}_p\rangle\!\rangle = 1$ and the conjugation properties (4.1), (4.3) hold, then $(\mathcal{H}_p, \langle\!\langle\cdot\,|\,\cdot\rangle\!\rangle_p)$ becomes isomorphic to the NS moduli $(\mathcal{V}_a, \langle\cdot\,|\,\cdot\rangle_a)$ with $a = \frac{Q}{2} + ip$.

Define now:

- a normal ordered exponentials:

$$
\mathsf{E}^{\alpha}(\sigma) \;=\; \mathrm{e}^{\frac{1}{2}\alpha\mathsf{q}}\,\mathrm{e}^{\alpha\varphi_<(\sigma)}\,\mathrm{e}^{\alpha\sigma\mathsf{p}}\,\mathrm{e}^{\alpha\varphi_>(\sigma)}\,\mathrm{e}^{\frac{1}{2}\alpha\mathsf{q}},
$$

  built from the chiral scalar in a way which assures their hermiticity for real α, and

- the screening charge:

$$
\mathsf{Q}(\sigma) \;=\; \int_{\sigma}^{\sigma+2\pi} dx\;\psi(x)\mathsf{E}^b(x).
$$

In the present paper we are interested in algebraic properties of normal ordered exponentials and screening charges. Therefore, we shall not discuss some of the properties – such as their domain or self-adjointnes issues – which would allow to establish them as true operators (see however a discussion on the parallel issues for the non-supersymmetric case in [11]).

Explicit calculations give:

$$
\begin{aligned}
[L_n, \mathsf{E}^{\alpha}(\sigma)] &= \mathrm{e}^{in\sigma}\left(-i\frac{d}{d\sigma} + n\Delta_\alpha\right)\mathsf{E}^{\alpha}(\sigma), \qquad \Delta_\alpha = \frac{1}{2}\alpha(Q-\alpha), \\
[L_n, \psi(\sigma)] &= \mathrm{e}^{in\sigma}\left(-i\frac{d}{d\sigma} + n\frac{1}{2}\right)\psi(\sigma),
\end{aligned}
\tag{4.4}
$$

and

$$
[S_k, \mathsf{E}^{\alpha}(\sigma)] \;=\; -i\alpha\,\mathrm{e}^{ik\sigma}\,\psi(\sigma)\,\mathsf{E}^{\alpha}(\sigma). \tag{4.5}
$$

Moreover, if we take

$$
Q = b + b^{-1}, \tag{4.6}
$$

then

$$
\Big[L_n, \psi(\sigma)\mathsf{E}^b(\sigma)\Big] = -i\frac{d}{d\sigma}\Big(\mathrm{e}^{in\sigma}\psi(\sigma)\mathsf{E}^b(\sigma)\Big) \tag{4.7}
$$

and

$$\left\{S_k, \psi(\sigma)\mathsf{E}^b(\sigma)\right\} \;=\; \frac{1}{b}\frac{d}{d\sigma}\left(\mathrm{e}^{ik\sigma}\,\mathsf{E}^b(\sigma)\right), \tag{4.8}$$

what shows that $\mathsf{Q}(\sigma)$ is a scalar under the transformations generated by L_n and S_k.

For real b the screening charge $\mathsf{Q}(\sigma)$ is hermitian, its square is therefore positive and $\left[\mathsf{Q}(\sigma)^2\right]^t$ may be uniquely defined for complex t. This motivates the following definition of an "even" and an "odd" complex powers of the screening charge: for $s \in \mathbb{C}\setminus\mathbb{Z}$:

$$(\mathsf{Q}(\sigma))^s_{\mathrm{e}} = \left(\mathsf{Q}^2(\sigma)\right)^{\frac{s}{2}}, \qquad (\mathsf{Q}(\sigma))^s_{\mathrm{o}} \;=\; \mathsf{Q}(\sigma)\left(\mathsf{Q}^2(\sigma)\right)^{\frac{s-1}{2}} \tag{4.9}$$

and

$$(\mathsf{Q}(\sigma))^s_{\mathrm{e}} = (\mathsf{Q}(\sigma))^s_{\mathrm{o}} = (\mathsf{Q}(\sigma))^s \qquad \text{for } s \in \mathbb{N}.$$

Relations (4.4) – (4.8) then imply that chiral fields

$$\mathsf{g}^{\alpha,\,\rho}_s(\sigma) \;=\; \mathsf{E}^{\alpha}(\sigma)\,(\mathsf{Q}(\sigma))^s_{\rho}\,, \qquad \rho = \mathrm{e},\ \mathrm{o},$$

transform covariantly under superconformal transformations.

As in the non-supersymmetric case we can define the Euclidean fields by analytic continuation to imaginary time

$$\mathsf{g}^{\alpha,\,\rho}_s(w) = \mathrm{e}^{\tau L_0}\,\mathsf{g}^{\alpha,\,\rho}_s(\sigma)\,\mathrm{e}^{-\tau L_0}, \qquad w = \tau + i\sigma. \tag{4.10}$$

The fields (4.10) on a complex cylinder parameterized by the coordinate w are related to the fields $\mathsf{g}^{\alpha,\,\rho}_s(z)$ on a complex plane $z = \mathrm{e}^w$ via

$$\mathsf{g}^{\alpha,\,\rho}_s(w) = z^{\Delta_\alpha}\,\mathsf{g}^{\alpha,\,\rho}_s(z), \qquad \Delta_\alpha = \frac{1}{2}\alpha(Q-\alpha).$$

Thanks to a simple dependence on q, the fields $\mathsf{g}^{\alpha,\,\rho}_s(w)$ have simple commutation properties with functions of p,

$$\mathsf{g}^{\alpha,\,\rho}_s(w) f(\mathsf{p}) = f\big(\mathsf{p} - i(\alpha + bs)\big)\mathsf{g}^{\alpha,\,\rho}_s(w).$$

This relation, the isomorphism $(\mathcal{H}_p, \langle\!\langle\cdot\,|\,\cdot\rangle\!\rangle_p) \simeq (\mathcal{V}_a, \langle\cdot\,|\,\cdot\rangle_{\frac{Q}{2}+ip})$ and equations (4.4) – (4.8) show that a restriction of $\mathsf{g}^{\alpha,\,\rho}_s(w)$ to $\mathcal{H}_p$ provides a realization of a (unnormalized) superconformal vertex operator ${}^{\mathsf{g}}V^{\rho}_{a_3 a_1}(\nu_\alpha|w)$ with $a_1 = \frac{Q}{2} + ip$ and $a_3 = a_1 + \alpha + bs$.

4.2 Matrix elements

For the chiral field on the complex z plane let us denote:

$$\begin{aligned}
\mathcal{M}(a_2, s|a_1) &\equiv \langle\!\langle\tilde{\nu}_q|\mathsf{g}^{a_2,\mathrm{e}}_s(1)\tilde{\nu}_p\rangle\!\rangle_q = \langle\nu_3|\,{}^{\mathsf{g}}V^{\mathrm{e}}_{a_3 a_1}(\nu_2|1)\nu_1\rangle_{a_3}\,,\\
\mathcal{M}^*(a_2, s|a_1) &\equiv \langle\!\langle\tilde{\nu}_q|\,*\mathsf{g}^{a_2,\mathrm{e}}_s(1)\tilde{\nu}_p\rangle\!\rangle_q = \langle\nu_3|\,{}^{\mathsf{g}}V^{\mathrm{e}}_{a_3 a_1}(*\nu_2|1)\nu_1\rangle_{a_3}\,,
\end{aligned} \tag{4.11}$$

where

$$*\mathsf{g}^{a,\mathrm{e}}_s(1) = \left\{S_{-1/2}, \mathsf{g}^{a,\mathrm{o}}_s(1)\right\}, \qquad p = \frac{iQ}{2} - ia_1, \qquad q = \frac{iQ}{2} - i(a_1 + a_2 + bs) \equiv \frac{iQ}{2} - ia_3.$$

To compute these matrix elements we shall use a strategy similar to the one applied by J. Teschner in calculating the matrix element of the chiral primary field in the non-supersymmetric CFT [11].

Introduce two auxiliary fields:

$$\begin{aligned} \mathsf{g}_0(z) &\equiv \mathsf{g}_0^{-b}(z) = \mathsf{E}^{-b}(z), \\ \mathsf{g}_1(z) &\equiv \mathsf{g}_1^{-b}(z) = \mathsf{E}^{-b}(z)\mathsf{Q}(z), \end{aligned}$$

together with their descendants $*\mathsf{g}_r(z),\ r = 0, 1$. We shall need the following simple matrix elements[4]:

$$\langle\!\langle \tilde{\nu}_{p+ib} | \mathsf{g}_0(1) \tilde{\nu}_p \rangle\!\rangle_{p+ib} \equiv \langle\!\langle p + ib | \mathsf{g}_0(1) | p \rangle\!\rangle = 1, \qquad\qquad \langle\!\langle p + ib | {*\mathsf{g}_0(1)} | {*p} \rangle\!\rangle \;=\; b\beta,$$

and

$$\langle\!\langle p | \mathsf{g}_1(1) | {*p} \rangle\!\rangle \;=\; \langle\!\langle p | {*\mathsf{g}_1(1)} | p \rangle\!\rangle \;=\; \frac{2\pi i \beta \, \mathrm{e}^{-i\pi b\beta}\, \Gamma(1+b^2)}{\Gamma(1+b\beta)\Gamma(1+b^2-b\beta)} \;\equiv\; \mathsf{M}_1(\beta), \tag{4.12}$$

with $\beta = \frac{Q}{2} + ip$ and where, in order to derive (4.12), we used (after a suitable deformation of the integration contour) the integral representation of the Euler beta function.

Consider now conformal blocks:

$$\begin{aligned} \Psi_r(z) &\equiv \langle\!\langle p_4^r |\ \mathsf{g}_r(z) \mathsf{g}_s^{a_2}(1)\ | p_1 \rangle\!\rangle, \\ \Psi_r^*(z) &\equiv \langle\!\langle p_4^r |\ {*\mathsf{g}_r(z)} \mathsf{g}_s^{a_2}(1)\ | p_1 \rangle\!\rangle, \end{aligned} \tag{4.13}$$

where $p_4^r = p_1 - i(a_2 + bs + b\,(r-1))$ and where we have suppressed the "parity" superscript ρ of $\mathsf{g}_s^{a_2,\rho}$ since if we require that $\Psi_r(z)$ and $\Psi_r^*(z)$ do not vanish identically it is uniquely determined by the parity of $\mathsf{g}_r(z)$ and $*\mathsf{g}_r(z)$. We shall now evaluate in two different ways the leading (most singular) terms in the expansion of $\Psi_r(z)$ and $\Psi_r^*(z)$ around $z = 1$, arriving at a recurrence relation for the matrix elements (4.11).

First of all, just from the definition of the operators $\mathsf{g}_s^a(z)$ and $*\mathsf{g}_s^a(z)$, one can compute leading terms in their operator product expansions. For instance:

$$\mathsf{g}_0(z)\mathsf{g}_s^a(1) \;=\; \mathsf{E}^{-b}(z)\mathsf{E}^a(1)\,(\mathsf{Q}(1))^s \;\simeq\; (z-1)^{ba}\mathsf{E}^{a-b}(1)\,(\mathsf{Q}(1))^s \;=\; (z-1)^{ba}\mathsf{g}_s^{a-b}(1),$$

or, writing $\mathsf{Q}(z) = \mathsf{Q}(1) + (\mathsf{Q}(z) - \mathsf{Q}(1))$,

$$\begin{aligned} \mathsf{g}_1(z)\mathsf{g}_s^a(1) \;&=\; \mathsf{E}^{-b}(z)\mathsf{Q}(1)\mathsf{E}^a(1)\,(\mathsf{Q}(1))^s + \mathsf{E}^{-b}(z)\,(\mathsf{Q}(z) - \mathsf{Q}(1))\,\mathsf{E}^a(1)\,(\mathsf{Q}(1))^s \\ &\simeq \mathrm{e}^{-i\pi ba}\mathsf{E}^{-b}(z)\mathsf{E}^a(1)\,(\mathsf{Q}(1))^{s+1} \;\simeq\; \mathrm{e}^{-i\pi ba}(z-1)^{ba}\mathsf{g}_{s+1}^{a-b}(1). \end{aligned}$$

[4] As before, we shall suppress the subscript q in the form $\langle\!\langle \cdot | \cdot \rangle\!\rangle_q$; since in the adapted notation it is the same as an argument of the "bra" it shouldn't result in any confusion.

Thus we have:

$$\begin{aligned} \Psi_0(z) &\underset{z\to 1}{\simeq} \mathcal{M}(a_2-b,s|p_1)\;(z-1)^{ba_2}, \\ \Psi_1(z) &\underset{z\to 1}{\simeq} \mathsf{e}^{-i\pi b a_2}\mathcal{M}(a_2-b,s+1|p_1)\;(z-1)^{ba_2}, \end{aligned} \tag{4.14}$$

and similarly:

$$\begin{aligned} \Psi_0^*(z) &\underset{z\to 1}{\simeq} -\frac{b}{a_2-b}\mathcal{M}^*(a_2-b,s|p_1)\;(z-1)^{ba_2}, \\ \Psi_1^*(z) &\underset{z\to 1}{\simeq} -\frac{b}{a_2-b}\mathsf{e}^{-i\pi b a_2}\mathcal{M}^*(a_2-b,s+1|p_1)\;(z-1)^{ba_2}. \end{aligned} \tag{4.15}$$

On the other hand, the Verma moduli $\mathcal{V}_{-b}$ is degenerate: the vector

$$\big(L_{-1}S_{-\frac{1}{2}}+b^2S_{-\frac{3}{2}}\big)\nu_{-b}$$

is null. Consequently, correlators containing any of the fields $\mathsf{g}_r(z)$ satisfy the corresponding null vector decoupling equations [1, 18]. Their forms[5] for the conformal blocks (4.13) are:

$$\begin{aligned} &\frac{1}{b^2}\frac{d^3\Psi_r(z)}{dz^3}+\frac{(2z-1)(2b^2-1)}{z(1-z)b^2}\frac{d^2\Psi_r(z)}{dz^2}+ \\ &\left(\frac{b^2+2\Delta_1}{z^2}+\frac{b^2+2\Delta_2}{(1-z)^2}+\frac{2-3b^2+2\Delta^{(r)}_{1+2-4}}{z(1-z)}\right)\frac{d\Psi_r(z)}{dz}+ \\ &\left(\frac{2\Delta_2(1+b^2)}{(1-z)^3}-\frac{2\Delta_1(1+b^2)}{z^3}+\frac{\Delta_{2-1}+(1-2z)(b^4+b^2(1/2-\Delta^{(r)}_{1+2-4})-\Delta_{1+2})}{z^2(1-z)^2}\right)\Psi_r(z)\;=\;0, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} &\frac{d^3\Psi_r^*(z)}{dz^3}+\frac{2(1-b^2)(1-2z)}{z(1-z)}\frac{d^2\Psi^*_r(z)}{dz^2}+ \\ &\left(\frac{b^4-b^2+2b^2\left(\Delta_1(1-z)+\Delta_2 z\right)}{z^2(1-z)^2}-\frac{5b^4+b^2(2\Delta_4^{(r)}-7)+2}{z(1-z)}\right)\frac{d\Psi^*_r(z)}{dz}+ \\ &b^4\left(\frac{3(\Delta_1-\Delta_2)+\left(\Delta_4^{(r)}+b^2-1\right)(1-2z)}{z^2(1-z)^2}+\frac{2\Delta_2 z-2\Delta_1(1-z)}{z^3(1-z)^3}\right)\Psi^*_r(z)\;=\;0, \end{aligned} \tag{4.17}$$

where $\Delta_4^{(r)}=\frac{1}{2}a_4^{(r)}\left(Q-a_4^{(r)}\right)$ with

$$a_4^{(r)}=a_1+a_2+b(s+r-1),$$

and $\Delta_{1+2}=\Delta_1+\Delta_2$, $\Delta^{(r)}_{1+2-4}=\Delta_1+\Delta_2-\Delta^{(r)}_4$ etc.

Let us start by analyzing the equation (4.16). According to [17, 18] its solution is of the form

$$\Psi_r(z)=z^{a_1 b}(1-z)^{a_2 b}F(z), \tag{4.18}$$

[5] A derivation can be found in e.g. [18, 19]

where $F(z)$ can be expressed as a linear combination of Dotsenko-Fatteev type integrals

$$I_{\alpha\beta}(z) = \left(1 - \frac{1}{2}\delta_{\alpha\beta}\right) \int\limits_{\mathcal{C}_\alpha} dt_1 \int\limits_{\mathcal{C}_\beta} dt_2 \, (t_1 t_2)^{A_r} \left[(t_1-1)(t_2-1)\right]^{B_r} \left[(t_1-z)(t_2-z)\right]^{C_r} |t_2 - t_1|^{2g} , \tag{4.19}$$

with the integration contours $\mathcal{C}_1 = (-\infty, 0],$, $\mathcal{C}_2 = [0,1]$, $\mathcal{C}_3 = [1,z]$, $\mathcal{C}_4 = [z,\infty)$ and

$$\begin{aligned} A_r &= -\frac{1}{2} + \frac{b(a_4^{(r)} + a_2 - a_1)}{2}, & B_r &= -\frac{1}{2} + \frac{b(a_4^{(r)} - a_2 + a_1)}{2}, \\ C_r &= \frac{1}{2} - \frac{b(a_4^{(r)} + a_2 + a_1)}{2} + b^2, & g &= -\frac{1}{2} - \frac{b^2}{2}. \end{aligned} \tag{4.20}$$

In order to decide which solution of the differential equation (4.16) corresponds to the block $\langle\!\langle p_4^r | \, \mathsf{g}_r(z) \mathsf{g}_s^{a_2}(1) \, | p_1 \rangle\!\rangle$ let us note that we can present it in a form of a power series in z around $z = \infty$ by inserting between the fields $\mathsf{g}_r(z)$ and $\mathsf{g}_s^{a_2}(1)$ the projection operator onto the highest weight state with the momentum $q = p_1 - i(\alpha_2 + bs)$ and its (normalized) NS descendants. The leading terms in these expansions read

$$\begin{aligned} \Psi_0(z) \; &\underset{z\to\infty}{\simeq} \; \langle\!\langle q - ib | \, \mathsf{g}_0(z) \, | q \rangle\!\rangle \, \langle\!\langle q | \, \mathsf{g}_s^{a_2}(1) \, | p_1 \rangle\!\rangle , \\ \Psi_1(z) \; &\underset{z\to\infty}{\simeq} \; \frac{1}{2\Delta_4} \, \langle\!\langle q | \, \mathsf{g}_1(z) S_{-\frac{1}{2}} \, | q \rangle\!\rangle \, \langle\!\langle q | \, S_{\frac{1}{2}} \mathsf{g}_s^{a_2}(1) \, | p_1 \rangle\!\rangle , \end{aligned}$$

so that, using (4.11) and (4.12), we have:

$$\begin{aligned} \Psi_0(z) \; &\underset{z\to\infty}{\simeq} \; \mathcal{M}(a_2, s|a_1) \, z^{b(a_4^{(0)}+b)}, \\ \Psi_1(z) \; &\underset{z\to\infty}{\simeq} \; \frac{\mathsf{M}_1(a_4^{(1)})}{2\Delta_4^{(1)}} \mathcal{M}^*(a_2, s|a_1) \, z^{b^2}. \end{aligned} \tag{4.21}$$

Monodromies (around $z = \infty$) of all the terms in the expansion of $\Psi_0(z)$ are equal and determined by (4.21); the same is also true for $\Psi_1(z)$. On the other hand, out of the integrals (4.19), $I_{22}(z), I_{24}(z)$ and $I_{44}(z)$ form (once multiplied by $z^{a_1 b}(1-z)^{a_2 b}$) a basis in the space of solutions of (4.16) with a monodromy matrix diagonal around $z = \infty$. In fact, from (4.19) we have:

$$\begin{aligned} I_{44}(z) \; &\underset{z\to\infty}{\simeq} \; z^{-b(a_1+a_2)+b(a_4^{(0)}+b)} \, \mathcal{I}_{44}^{(\infty)} \left(1 + \mathcal{O}(z^{-1})\right) , \\ I_{24}(z) \; &\underset{z\to\infty}{\simeq} \; z^{-b(a_1+a_2)+b^2} \, \mathcal{I}_{24}^{(\infty)} \left(1 + \mathcal{O}(z^{-1})\right) , \end{aligned} \tag{4.22}$$

where:

$$\begin{aligned} \mathcal{I}_{44}^{(\infty)} &= \frac{\Gamma(2g)\Gamma(-1-A_0-B_0-C_0-g)\Gamma(-1-A_0-B_0-C_0-2g)}{\Gamma(g)} \\ &\quad \times \frac{\Gamma(1+C_0)\Gamma(1+C_0+g)}{\Gamma(-A_0-B_0)\Gamma(-A_0-B_0-g)}, \end{aligned} \tag{4.23}$$

$$\mathcal{I}_{24}^{(\infty)} = \Gamma(1+A_1)\Gamma(-1-A_1-B_1-C_1-2g) \, \frac{\Gamma(1+B_1)\Gamma(1+C_1)}{\Gamma(2+A_1+B_1)\Gamma(-A_1-B_1-2g)},$$

and, comparing (4.18) and (4.22) with (4.21) we get:

$$\begin{aligned} \Psi_0(z) &= \frac{1}{\mathcal{I}_{44}^{(\infty)}}\,\mathcal{M}(a_2,s|a_1)\,z^{ba_1}(z-1)^{ba_2}\,I_{44}(z), \\ \Psi_1(z) &= \frac{\mathsf{M}_1(a_4^{(1)})}{2\Delta_4^{(1)}\,\mathcal{I}_{24}^{(\infty)}}\,\mathcal{M}^*(a_2,s|a_1)\,z^{ba_1}(z-1)^{ba_2}\,I_{24}(z). \end{aligned} \tag{4.24}$$

Another basis in the space of solutions of (4.16), with a monodromy matrix diagonal around $z = 1$, is formed by $I_{11}(z), I_{13}(z)$ and $I_{33}(z)$. Obviously these bases are linearly related,

$$I_\imath = \sum_\jmath M_{\imath\jmath}\,I_\jmath, \qquad \imath \in \{22, 24, 44\}, \quad \jmath \in \{11, 13, 33\},$$

and the matrix $M_{\imath\jmath}$ is known, see [18, 25]. Using this and noticing that the leading contribution in the $z \to 1$ limit is given by the term proportional to $I_{11}(z)$ we have:

$$\begin{aligned} I_{44}(z) \underset{z\to 1}{\simeq} \mathcal{I}_{44}^{(1)} &= \frac{\Gamma(2g)\Gamma(-1-A_0-B_0-C_0-g)\Gamma(-1-A_0-B_0-C_0-2g)}{\Gamma(g)} \\ &\times \frac{\Gamma(1+B_0+C_0)\Gamma(1+B_0+C_0+g))}{\Gamma(-A_0)\Gamma(-A_0-g)}, \\ I_{24}(z) \underset{z\to 1}{\simeq} \mathcal{I}_{24}^{(1)} &= \Gamma(1+A_1)\Gamma(-1-A_1-B_1-C_1-2g) \\ &\times \frac{\Gamma(1-g)\Gamma(1+B_1+C_1)\Gamma(1+B_1+C_1+g)}{\Gamma(1-2g)\Gamma(-A_1-g)\Gamma(2+A_1+B_1+C_1+g)}. \end{aligned} \tag{4.25}$$

Substituting this result into (4.24) we get:

$$\begin{aligned} \Psi_0(z) &\underset{z\to 1}{\simeq} \frac{\mathcal{I}_{44}^{(1)}}{\mathcal{I}_{44}^{(\infty)}}\,\mathcal{M}(a_2,s|a_1)\,(z-1)^{ba_2}, \\ \Psi_1(z) &\underset{z\to 1}{\simeq} \frac{\mathsf{M}_1(a_4)\mathcal{I}_{24}^{(1)}}{2\Delta_4^{(1)}\,\mathcal{I}_{24}^{(\infty)}}\,\mathcal{M}^*(a_2,s|a_1)\,(z-1)^{ba_2}. \end{aligned} \tag{4.26}$$

Comparing (4.26) with (4.14) and using (4.12) we arrive at the recurrence relations of the form:

$$\begin{aligned} \mathcal{M}(a_2-b,s|a_1) &= \frac{\mathcal{I}_{44}^{(1)}}{\mathcal{I}_{44}^{(\infty)}}\,\mathcal{M}(a_2,s|a_1), \\ \mathcal{M}(a_2-b,s+1|a_1) &= \mathrm{e}^{i\pi b(a_2-a_4^{(1)})}\,\frac{\mathcal{I}_{24}^{(1)}}{\mathcal{I}_{24}^{(\infty)}}\,\frac{2\pi i a_4\Gamma(1+b^2)}{2\Delta_4^{(1)}\,\Gamma(1+ba_4^{(1)})\Gamma(1+b^2-ba_4^{(1)})}\,\mathcal{M}^*(a_2,s|a_1). \end{aligned}$$

Using equations (4.23), (4.25), (4.20) and the relations satisfied by the Barnes gamma

function (Appendix A) we thus have:

$$\frac{\mathcal{M}(a_2-b,s|a_1)}{\mathcal{M}(a_2,s|a_1)} = \frac{\Gamma(1+B_0+C_0)\Gamma(1+B_0+C_0+g)\Gamma(-A_0-B_0)\Gamma(-A_0-B_0-g)}{\Gamma(-A_0)\Gamma(-A_0-g)\Gamma(1+C_0)\Gamma(1+C_0+g)}$$

$$= \left[\frac{\Gamma_{\rm NS}(2Q-2a_1-2a_2-bs)\Gamma_{\rm NS}(Q-2a_2-bs)}{\Gamma_{\rm NS}(Q-2a_2)\Gamma_{\rm NS}\left(2Q-2a_1-2a_2-2bs\right)}\right]^{-1} \tag{4.27}$$

$$\times\ \frac{\Gamma_{\rm NS}(2Q-2a_1-2(a_2-b)-bs)\Gamma_{\rm NS}(Q-2(a_2-b)-bs)}{\Gamma_{\rm NS}(Q-2(a_2-b))\Gamma_{\rm NS}\left(2Q-2a_1-2(a_2-b)-2bs\right)},$$

and similarly:

$$\frac{\mathcal{M}(a_2-b,s+1|a_1)}{\mathcal{M}^*(a_2,s|a_1)} = i\mathrm{e}^{-i\pi b(a_1+bs)}\,\frac{1}{2}\,\Gamma\left(\frac{bQ}{2}\right)\,b^{-\frac{bQ}{2}}$$

$$\times\ \left[\frac{\Gamma_{\rm R}\left(Q-a_2-bs\right)\Gamma_{\rm R}\left(2a_1+bs\right)\Gamma_{\rm R}\left(Q+bs\right)\Gamma_{\rm R}\left(2Q-2a_{1+2}-bs\right)}{\Gamma_{\rm NS}(Q-2a_2)}\right]^{-1} \tag{4.28}$$

$$\times\ \frac{\Gamma_{\rm NS}\left(Q-2a_2-bs+b\right)\Gamma_{\rm NS}\left(2a_1+bs+b\right)\Gamma_{\rm NS}\left(Q+bs+b\right)\Gamma_{\rm NS}\left(2Q-2a_{1+2}-bs+b\right)}{\Gamma_{\rm NS}(Q-2a_2+2b)},$$

where $a_{1+2}\equiv a_1+a_2$.

To arrive at a second set of recursion relations we repeat the same steps for the blocks $\Psi^*_r(z)$: calculating from (4.13) the leading behavior of $\Psi^*_r(z)$ for $z\to\infty$ we identify the appropriate solutions of the differential equations (4.17), then we express them in the basis given by the functions with the monodromy matrix diagonal around $z=1$ and compare the result with the formula (4.15). This yields:

$$\frac{\mathcal{M}^*(a_2-b,s|a_1)}{\mathcal{M}^*(a_2,s|a_1)} = \left[\frac{\Gamma_{\rm R}(Q-2a_2-bs)\Gamma_{\rm R}(2Q-2a_1-2a_2-bs)}{\Gamma_{\rm NS}(Q-2a_2)\Gamma_{\rm NS}(2Q-2a_1-2a_2-2bs)}\right]^{-1} \tag{4.29}$$

$$\times\ \frac{\Gamma_{\rm R}(Q-2(a_2-b)-bs)\Gamma_{\rm R}(2Q-2a_1-2(a_2-b)-bs)}{\Gamma_{\rm NS}(Q-2(a_2-b))\Gamma_{\rm NS}(2Q-2a_1-2(a_2-b)-2bs)},$$

and

$$\frac{\mathcal{M}^*(a_2-b,s+1|a_1)}{\mathcal{M}(a_2,s|a_1)} = -\,i\mathrm{e}^{-i\pi b(a_1+bs)}\,\Gamma\left(\frac{bQ}{2}\right)\,b^{-\frac{bQ}{2}}$$

$$\times\ \left[\frac{\Gamma_{\rm NS}\left(Q-a_2-bs\right)\Gamma_{\rm NS}\left(2a_1+bs\right)\Gamma_{\rm NS}\left(Q+bs\right)\Gamma_{\rm NS}\left(2Q-2a_1-2a_2-bs\right)}{\Gamma_{\rm NS}(Q-2a_2)}\right]^{-1} \tag{4.30}$$

$$\times\ \frac{\Gamma_{\rm R}\left(Q-2a_2-bs+b\right)\Gamma_{\rm R}\left(2a_1+bs+b\right)\Gamma_{\rm R}\left(Q+bs+b\right)\Gamma_{\rm R}\left(2Q-2a_1-2a_2-bs+b\right)}{\Gamma_{\rm NS}(Q-2a_2+2b)}.$$

The solution of recurrence relations (4.27) – (4.30) is not unique. Notice however that — exactly as in the non-supersymmetric case — we can repeat the construction above with b replaced by b^{-1}. Moreover, direct calculation (essentially the same that leads to (4.12)) gives:

$$\mathcal{M}(a_2,0|a_1) = 1 \tag{4.31}$$

and

$$\mathcal{M}^*(a_2,1|a_1) \;=\; 2\pi i a_1\,\mathrm{e}^{-\pi i b a_1}\,\frac{\Gamma(1-ba_2)}{\Gamma(1+ba_1)\Gamma(1-ba_1-ba_2)}. \tag{4.32}$$

If b is real and irrational, relations (4.27) – (4.32) together with the "dual" ($b \to b^{-1}$) ones uniquely determine the the matrix elements of the $N=1$ supersymmetric chiral vertex operators to be:

$$\begin{aligned}\mathcal{M}(a_2,s|a_1) = {}& \left[\frac{1}{2}\Gamma\left(\frac{bQ}{2}\right)b^{-\frac{bQ}{2}}\right]^{\frac{a_3-a_1-a_2}{b}}\mathrm{e}^{\frac{i\pi}{2}(a_3-a_2-a_1)(Q-a_3+a_2-a_1)} \\ & \times\frac{\Gamma_{\mathrm{NS}}\left(Q+a_{1-2-3}\right)\Gamma_{\mathrm{NS}}\left(a_{1+3-2}\right)\Gamma_{\mathrm{NS}}\left(Q+a_{3-1-2}\right)\Gamma_{\mathrm{NS}}\left(2Q-a_{1+2+3}\right)}{\Gamma_{\mathrm{NS}}(Q)\Gamma_{\mathrm{NS}}(2a_1)\Gamma_{\mathrm{NS}}(Q-2a_2)\Gamma_{\mathrm{NS}}(2Q-2a_3)}\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}\mathcal{M}^*(a_2,s|a_1) = {}& 2\left[\frac{1}{2}\Gamma\left(\frac{bQ}{2}\right)b^{-\frac{bQ}{2}}\right]^{\frac{a_3-a_1-a_2}{b}}\mathrm{e}^{\frac{i\pi}{2}(a_3-a_2-a_1)(Q-a_3+a_2-a_1)} \\ & \times\frac{\Gamma_{\mathrm{R}}\left(Q+a_{1-2-3}\right)\Gamma_{\mathrm{R}}\left(a_{1+3-2}\right)\Gamma_{\mathrm{R}}\left(Q+a_{3-1-2}\right)\Gamma_{\mathrm{R}}\left(2Q-a_{1+2+3}\right)}{\Gamma_{\mathrm{NS}}(Q)\Gamma_{\mathrm{NS}}(2a_1)\Gamma_{\mathrm{NS}}(Q-2a_2)\Gamma_{\mathrm{NS}}(2Q-2a_3)},\end{aligned} \tag{4.34}$$

where we have denoted

$$a_3 = a_1 + a_2 + bs.$$

4.3 Braiding relations

Thanks to the relation between chiral fields $\mathsf{g}_s^{\alpha,\,\rho}(z)$ and the (unnormalized) vertex operators ${}^{\mathsf{g}}\mathsf{V}^{\rho}_{a_3a_1}(\nu_\alpha|z)$ the form of the braiding matrix appearing in (3.11) can be derived by studying an exchange relation for the chiral fields.

Assume that for the chiral fields on a complex w cylinder at $\tau=0$, i.e. $\mathsf{g}_s^{\alpha,\,\rho}(\sigma)$, there exist an integral kernel B such that the identity

$$\mathsf{g}_{s_2}^{\alpha_2,\rho}(\sigma_2)\mathsf{g}_{s_1}^{\alpha_1,\eta}(\sigma_1) = \int d\mu(t_1,t_2)\sum_{\lambda,\delta=\,\mathrm{e,o}} B^{\epsilon}\big(\alpha_1,\alpha_2;{}^{t_1\,t_2}_{s_2\,s_1}\big)^{\rho\eta}{}_{\lambda\delta}\;\mathsf{g}_{t_1}^{\alpha_1,\lambda}(\sigma_1)\mathsf{g}_{t_2}^{\alpha_2,\delta}(\sigma_2), \tag{4.35}$$

with $\epsilon = \mathrm{sign}(\sigma_2-\sigma_1)$ and the integration measure to be specified later, holds.

Since the parity of a product of chiral fields does not depend on their order,

$$(-1)^{|\rho|+|\eta|} = (-1)^{|\lambda|+|\delta|}$$

where $|\mathrm{e}|=0$ and $|\mathrm{o}|=1$, we can discuss the "even", $(-1)^{|\rho|+|\eta|} = (-1)^{|\lambda|+|\delta|} = 1$, and the "odd", $(-1)^{|\rho|+|\eta|} = (-1)^{|\lambda|+|\delta|} = -1$, cases separately. Introducing a shorthand notation

$$\left[B^{\mathrm{e},\epsilon}\right]^{\rho}{}_{\eta} \equiv B^{\epsilon}\big(\alpha_1,\alpha_2;{}^{t_1\,t_2}_{s_2\,s_1}\big)^{\rho\rho}{}_{\eta\eta}\,, \qquad \left[B^{\mathrm{o},\epsilon}\right]^{\rho}{}_{\eta} \equiv B^{\epsilon}\big(\alpha_1,\alpha_2;{}^{t_1\,t_2}_{s_2\,s_1}\big)^{\rho\bar\rho}{}_{\eta\bar\eta}\,, \tag{4.36}$$

with $\bar{\rm e} = {\rm o}$, $\bar{\rm o} = {\rm e}$, we can write (4.35) in the form

$$
\begin{aligned}
\mathsf{g}_{s_2}^{\alpha_2,\rho}(\sigma_2)\mathsf{g}_{s_1}^{\alpha_1,\rho}(\sigma_1) &= \sum_{\eta={\rm e,o}} \int d\mu(t_1,t_2)\, \left[B^{{\rm e},\epsilon}\right]^{\rho}_{\ \eta}\, \mathsf{g}_{t_1}^{\alpha_1,\eta}(\sigma_1)\mathsf{g}_{t_2}^{\alpha_2,\eta}(\sigma_2), \\
\mathsf{g}_{s_2}^{\alpha_2,\rho}(\sigma_2)\mathsf{g}_{s_1}^{\alpha_1,\bar\rho}(\sigma_1) &= \sum_{\eta={\rm e,o}} \int d\mu(t_1,t_2)\, \left[B^{{\rm o},\epsilon}\right]^{\rho}_{\ \eta}\, \mathsf{g}_{t_1}^{\alpha_1,\eta}(\sigma_1)\mathsf{g}_{t_2}^{\alpha_2,\bar\eta}(\sigma_2), \qquad \rho = {\rm e,o},.
\end{aligned}
\tag{4.37}
$$

Our strategy in calculating the matrix B is again a suitable extension of the Teschner's technique. Suppose that $\sigma_2 > \sigma_1$, let $I = [\sigma_1,\sigma_2],\ I_c = [\sigma_2, \sigma_1+2\pi],\ I' = [\sigma_1+2\pi, \sigma_2+2\pi]$ and define:

$$
\begin{aligned}
\mathsf{Q}_{\rm I} &= \int_I dx\ \mathsf{E}^b(x)\psi(x), \\
\mathsf{Q}_{\rm I}^c &= \int_{I^c} dx\ \mathsf{E}^b(x)\psi(x), \\
\mathsf{Q}_{\rm I}' &= \int_{I'} dx\ \mathsf{E}^b(x)\psi(x) \ = \ -{\rm e}^{i\pi b^2}\, {\rm e}^{2\pi b \mathsf{p}}\, \mathsf{Q}_{\rm I}.
\end{aligned}
\tag{4.38}
$$

Since

$$
\mathsf{E}^\alpha(x)\mathsf{E}^\beta(y) = {\rm e}^{-i\pi\alpha\beta\,{\rm sign}(x-y)}\, \mathsf{E}^\beta(y)\mathsf{E}^\alpha(x),
\tag{4.39}
$$

we get:

$$
\begin{aligned}
\mathsf{E}^{\alpha_1}(\sigma_1)\mathsf{E}^{\alpha_2}(\sigma_2) &= {\rm e}^{i\pi\alpha_1\alpha_2}\mathsf{E}^{\alpha_2}(\sigma_2)\mathsf{E}^{\alpha_1}(\sigma_1), \\
\mathsf{Q}(\sigma_2)\mathsf{E}^{\alpha_1}(\sigma_1) &= \left(\mathsf{Q}_{\rm I}^c + \mathsf{Q}_{\rm I}'\right)\mathsf{E}^{\alpha_1}(\sigma_1) \ = \ \mathsf{E}^{\alpha_1}(\sigma_1)\left({\rm e}^{-i\pi b\alpha_1}\mathsf{Q}_{\rm I}^c + {\rm e}^{-3i\pi b\alpha_1}\mathsf{Q}_{\rm I}'\right), \\
\mathsf{Q}(\sigma_1)\mathsf{E}^{\alpha_2}(\sigma_2) &= \left(\mathsf{Q}_{\rm I}^c + \mathsf{Q}_{\rm I}\right)\mathsf{E}^{\alpha_2}(\sigma_2) \ = \ \mathsf{E}^{\alpha_2}(\sigma_2)\left({\rm e}^{-i\pi b\alpha_2}\mathsf{Q}_{\rm I}^c + {\rm e}^{i\pi b\alpha_2}\mathsf{Q}_{\rm I}\right).
\end{aligned}
$$

We can thus write:

$$
\begin{aligned}
\mathsf{g}_{s_2}^{\alpha_2,\rho}(\sigma_2)\mathsf{g}_{s_1}^{\alpha_1,\eta}(\sigma_1) &= \mathsf{E}^{\alpha_2}(\sigma_2)\mathsf{E}^{\alpha_1}(\sigma_1)\ {\rm e}^{-i\pi b s_2\alpha_1}\left(\mathsf{Q}_{\rm I}^c + {\rm e}^{-2i\pi b\alpha_1}\mathsf{Q}_{\rm I}'\right)^{s_2}_{\rho}\left(\mathsf{Q}_{\rm I}^c + \mathsf{Q}_{\rm I}\right)^{s_1}_{\eta}, \\
\mathsf{g}_{t_1}^{\alpha_1,\lambda}(\sigma_1)\mathsf{g}_{t_2}^{\alpha_2,\delta}(\sigma_2) &= \mathsf{E}^{\alpha_2}(\sigma_2)\mathsf{E}^{\alpha_1}(\sigma_1)\ {\rm e}^{i\pi\alpha_1\alpha_2 - i\pi b t_1\alpha_2}\left(\mathsf{Q}_{\rm I}^c + {\rm e}^{2i\pi b\alpha_2}\mathsf{Q}_{\rm I}\right)^{t_1}_{\lambda}\left(\mathsf{Q}_{\rm I}^c + \mathsf{Q}_{\rm I}'\right)^{t_2}_{\delta}.
\end{aligned}
\tag{4.40}
$$

The meaning of the r.h.s. of (4.40) is clear for natural s_k and $t_k,\ k = 1,2$. Notice however that for real b and purely imaginary α_1 and α_2 the hermiticity of $\mathsf{Q}_{\rm I}, \mathsf{Q}_{\rm I}^c$ and $\mathsf{Q}_{\rm I}'$ implies a hermiticity of their (multiplied by real numbers) sums. The "even" and "odd" powers on the r.h.s. are thus unambiguously defined (and thus the relation (4.40) is valid) also for complex s_k and t_k. Moreover, for s_k and t_k being purely imaginary the operators on the r.h.s. of (4.40) are (formally) unitary. We thus take

$$
\alpha_k, s_k, t_k \in i\mathbb{R}, \qquad k = 1,2.
\tag{4.41}
$$

It follows from (4.39) that the operators (4.38) satisfy a Weyl-type algebra:

$$\mathsf{Q}_{\mathrm{I}}\,\mathsf{Q}_{\mathrm{I}}^{c} \;=\; -\,\mathrm{e}^{i\pi b^2}\,\mathsf{Q}_{\mathrm{I}}^{c}\,\mathsf{Q}_{\mathrm{I}} \;\equiv\; -q\,\mathsf{Q}_{\mathrm{I}}^{c}\,\mathsf{Q}_{\mathrm{I}}, \qquad \mathsf{Q}_{\mathrm{I}}^{c}\,\mathsf{Q}_{\mathrm{I}}' \;=\; -q\,\mathsf{Q}_{\mathrm{I}}'\,\mathsf{Q}_{\mathrm{I}}^{c}, \qquad \mathsf{Q}_{\mathrm{I}}\,\mathsf{Q}_{\mathrm{I}}' \;=\; q^2\,\mathsf{Q}_{\mathrm{I}}'\,\mathsf{Q}_{\mathrm{I}}. \tag{4.42}$$

We can thus represent them in a form (see Appendix B for a derivation and a clarification on the 2×2 matrix structure):

$$\begin{aligned} \mathsf{Q}_{\mathrm{I}}^{c} &= \tau_1\,\mathrm{e}^{b\mathsf{x}}\,\mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}}, \\ \mathsf{Q}_{\mathrm{I}} &= \tau_2\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\,\mathrm{e}^{-\pi b\mathsf{p}}\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\,\mathrm{e}^{\frac{1}{2}i\pi b\mathsf{t}}, \end{aligned} \tag{4.43}$$

where $\tau_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\tau_2 = \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$ and the operators p, x and t satisfy commutation relations

$$[\mathsf{p},\mathsf{x}] \;=\; -i, \qquad [\mathsf{p},\mathsf{t}] \;=\; [\mathsf{x},\mathsf{t}] \;=\; 0, \tag{4.44}$$

together with a conjugation properties $\mathsf{x}^\dagger = \mathsf{x}^\dagger$, $\mathsf{t}^\dagger = -\mathsf{t}$.

The representation (4.43) and relations satisfied by special functions G_{NS}, G_{R} allow to arrange operators that appear on the r.h.s. of (4.40) in a "normal ordered form", with the operator x on the left and the operator p on the right. To this end notice that:

$$\mathsf{Q}_{\mathrm{I}}^{c} + \mathsf{Q}_{\mathrm{I}} = \mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}}\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}} \begin{pmatrix} 0 & 1 + i\mathrm{e}^{-\pi b(\mathsf{p}-i\mathsf{t})} \\ 1 - i\mathrm{e}^{-\pi b(\mathsf{p}-i\mathsf{t})} & 0 \end{pmatrix} \mathrm{e}^{\frac{1}{2}b\mathsf{x}},$$

and since

$$G_{\mathrm{NS}}(z+b) \;=\; \left(1+\mathrm{e}^{i\pi bz}\right) G_{\mathrm{R}}(z), \qquad G_{\mathrm{R}}(z+b) \;=\; \left(1-\mathrm{e}^{i\pi bz}\right) G_{\mathrm{NS}}(z),$$

we can write:

$$\begin{aligned} &\begin{pmatrix} 0 & 1 + i\mathrm{e}^{-\pi b(\mathsf{p}-i\mathsf{t})} \\ 1 - i\mathrm{e}^{-\pi b(\mathsf{p}-i\mathsf{t})} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & G_{\mathrm{NS}}(i\mathsf{p}+\mathsf{t}+\frac{1}{2b}+b) \\ G_{\mathrm{R}}(i\mathsf{p}+\mathsf{t}+\frac{1}{2b}+b) & 0 \end{pmatrix} \begin{pmatrix} G_{\mathrm{NS}}(i\mathsf{p}+\mathsf{t}+\frac{1}{2b}) & 0 \\ 0 & G_{\mathrm{R}}(i\mathsf{p}+\mathsf{t}+\frac{1}{2b}) \end{pmatrix}^{-1}. \end{aligned}$$

From (4.44) we see that for an analytic function f :

$$\mathrm{e}^{\alpha\mathsf{x}} f(\mathsf{p},\mathsf{t})\mathrm{e}^{-\alpha\mathsf{x}} \;=\; f(\mathsf{p}+i\alpha,\mathsf{t}),$$

so that

$$\mathrm{e}^{\frac{1}{2}b\mathsf{x}} \begin{pmatrix} 0 & 1 + i\mathrm{e}^{-\pi b(\mathsf{p}-i\mathsf{t})} \\ 1 - i\mathrm{e}^{-\pi b(\mathsf{p}-i\mathsf{t})} & 0 \end{pmatrix} \mathrm{e}^{\frac{1}{2}b\mathsf{x}} = \mathbb{G}_b\left(i\mathsf{p}+\mathsf{t}+\tfrac{Q}{2}\right) \begin{pmatrix} 0 & \mathrm{e}^{b\mathsf{x}} \\ \mathrm{e}^{b\mathsf{x}} & 0 \end{pmatrix} \mathbb{G}_b\left(i\mathsf{p}+\mathsf{t}+\tfrac{Q}{2}\right)^{-1},$$

where we denoted

$$\mathbb{G}_b\,(z) \;=\; \begin{pmatrix} G_{\mathrm{NS}}(z) & 0 \\ 0 & G_{\mathrm{R}}(z) \end{pmatrix}.$$

Our definition of "even" and "odd" complex powers, (4.9), thus gives:

$$
\begin{aligned}
(\mathsf{Q}_{\mathrm{I}}^c + \mathsf{Q}_{\mathrm{I}})_\rho^{s_1} &= \mathrm{e}^{-\frac{1}{2} i\pi b s_1 \mathsf{t}}\, \mathbb{G}_b\left(i\mathsf{p} + \mathsf{t} + \tfrac{Q}{2}\right) \mathrm{e}^{b s_1 \mathsf{x}}\, \mathbf{1}_\rho\, \mathbb{G}_b\left(i\mathsf{p} + \mathsf{t} + \tfrac{Q}{2}\right)^{-1} \\
&= \mathrm{e}^{b s_1 \mathsf{x}}\, \mathrm{e}^{-\frac{1}{2} i\pi b s_1 \mathsf{t}}\, \mathbb{G}_b\left(i\mathsf{p} + \mathsf{t} + b s_1 + \tfrac{Q}{2}\right) \mathbf{1}_\rho\, \mathbb{G}_b\left(i\mathsf{p} + \mathsf{t} + \tfrac{Q}{2}\right)^{-1},
\end{aligned}
$$

with $\mathbf{1}_{\mathrm{e}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\mathbf{1}_{\mathrm{o}} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Similarly:

$$
(\mathsf{Q}_{\mathrm{I}}^c + \mathsf{Q}_{\mathrm{I}}')_\lambda^{t_2} = \mathrm{e}^{b t_2 \mathsf{x}}\, \mathrm{e}^{-\frac{1}{2} i\pi b t_2 \mathsf{t}}\, \mathbb{G}_b\left(-i\mathsf{p} + \mathsf{t} - b t_2 + \tfrac{Q}{2}\right)^{-1} \mathbf{1}_\lambda\, \mathbb{G}_b\left(-i\mathsf{p} + \mathsf{t} + \tfrac{Q}{2}\right),
$$

and

$$
\begin{aligned}
(\mathsf{Q}_{\mathrm{I}}^c + \mathrm{e}^{-2i\pi b\alpha_1} \mathsf{Q}_{\mathrm{I}}')_\rho^{s_2} (\mathsf{Q}_{\mathrm{I}}^c + \mathsf{Q}_{\mathrm{I}})_\eta^{s_1} \;&=\; \mathrm{e}^{b(s_1+s_2)\mathsf{x}}\, \mathrm{e}^{-\frac{1}{2} i\pi b (s_1+s_2)\mathsf{t}} \\
\times\; \mathbb{G}_b&\left(-i\mathsf{p} + \mathsf{t} - 2\alpha_1 - b s_2 - b s_2 + \tfrac{Q}{2}\right)^{-1} \mathbf{1}_\rho\, \mathbb{G}_b\left(-i\mathsf{p} + \mathsf{t} - 2\alpha_1 - b s_1 + \tfrac{Q}{2}\right) \\
\times\; \mathbb{G}_b&\left(i\mathsf{p} + \mathsf{t} + b s_1 + \tfrac{Q}{2}\right) \mathbf{1}_\eta\, \mathbb{G}_b\left(i\mathsf{p} + \mathsf{t} + \tfrac{Q}{2}\right)^{-1},
\end{aligned}
$$

$$
\begin{aligned}
(\mathsf{Q}_{\mathrm{I}}^c + \mathrm{e}^{2i\pi b\alpha_2} \mathsf{Q}_{\mathrm{I}})_\lambda^{t_1} (\mathsf{Q}_{\mathrm{I}}^c + \mathsf{Q}_{\mathrm{I}}')_\delta^{t_2} \;&=\; \mathrm{e}^{b(t_1+t_2)\mathsf{x}}\, \mathrm{e}^{-\frac{1}{2} i\pi b (t_1+t_2)\mathsf{t}} \\
\times\; \mathbb{G}_b&\left(i\mathsf{p} + \mathsf{t} + 2\alpha_2 + b t_1 + b t_2 + \tfrac{Q}{2}\right) \mathbf{1}_\lambda\, \mathbb{G}_b\left(i\mathsf{p} + \mathsf{t} + 2\alpha_2 + b t_2 + \tfrac{Q}{2}\right)^{-1} \\
\times\; \mathbb{G}_b&\left(-i\mathsf{p} + \mathsf{t} - b t_2 + \tfrac{Q}{2}\right)^{-1} \mathbf{1}_\delta\, \mathbb{G}_b\left(-i\mathsf{p} + \mathsf{t} + \tfrac{Q}{2}\right).
\end{aligned}
$$

Since p and t commute we can evaluate the action of both sides of (4.37) on a common eigenstate of the momentum p (with an eigenvalue $p_1 \in \mathbb{R}$) and t (with an eigenvalue $\tau \in i\mathbb{R}$). Conservation of the momentum gives

$$
t_1 + t_2 \;=\; s_1 + s_2 \;\stackrel{\text{def}}{=}\; s
$$

and the integration measure thus reads $d\mu(t_1, t_2) = \delta(t_1 + t_2 - s) d\mu(t_1) d\mu(t_2)$. Define:

$$
\begin{aligned}
&A_1 = p_1 - 2i\alpha_1 - ibs, \;\; B_1 \;=\; -p_1, \;\; C_1 \;=\; i(\alpha_1 - Q/2), \;\; p_s \;=\; p_1 - i(\alpha_1 + b s_1), \\
&A_2 = p_1 - 2i\alpha_2 - ibs, \;\; B_2 \;=\; -p_1, \;\; C_2 \;=\; i(\alpha_2 - Q/2), \;\; p_u \;=\; p_1 - i(\alpha_2 + b t_2).
\end{aligned} \tag{4.45}
$$

It turns out to be convenient to express the functions $G_{\mathrm{NS,R}}$ through their "cousins" $S_{\mathrm{NS,R}}$, see Appendix A. Denoting

$$
\mathbb{S}_b(z) \;=\; \begin{pmatrix} S_{\mathrm{NS}}(z) & 0 \\ 0 & S_{\mathrm{R}}(z) \end{pmatrix}
$$

and using the reflection property $\mathbb{S}_b(z)\, \mathbb{S}_b(Q - z) = \mathbf{1}_{\mathrm{e}}$ we get:

$$
\begin{aligned}
&\mathsf{g}_{s_2}^{\alpha_2,\rho}(\sigma_2) \mathsf{g}_{s_1}^{\alpha_1,\eta}(\sigma_1)\Big|_{\substack{\mathsf{p}=p_1 \\ \mathsf{t}=\tau}} \;=\; \mathsf{E}^{\alpha_2}(\sigma_2) \mathsf{E}^{\alpha_1}(\sigma_1)\; \mathrm{e}^{bs\mathsf{x}}\; \mathrm{e}^{\frac{\pi b}{2} p_1 (s_2 - s_1) - 2i\pi b s_2 \alpha_1 + \frac{i\pi b^2}{2} s_1^2 - \frac{i\pi b^2}{4} s^2} \\
&\;\times \mathbb{S}_b\Big(\tfrac{Q}{2} + iA_1 - \tau\Big) F_\rho^T\, \mathbb{S}_b^{-1}(Q - iC_1 + ip_s - \tau)\, \mathbb{S}_b^{-1}(Q - iC_1 - ip_s - \tau) F_\eta\, \mathbb{S}_b\Big(\tfrac{Q}{2} + iB_1 - \tau\Big), \\
&\mathsf{g}_{t_1}^{\alpha_1,\lambda}(\sigma_1) \mathsf{g}_{t_2}^{\alpha_2,\delta}(\sigma_2)\Big|_{\substack{\mathsf{p}=p_1 \\ \mathsf{t}=\tau}} \;=\; \mathsf{E}^{\alpha_2}(\sigma_2) \mathsf{E}^{\alpha_1}(\sigma_1)\; \mathrm{e}^{bs\mathsf{x}}\; \mathrm{e}^{\frac{\pi b}{2} p_1 (s_2 - s_1) + i\pi \alpha_1 \alpha_2 - \frac{i\pi b^2}{2} t_2^2 + \frac{i\pi b^2}{4} s^2} \\
&\;\times \mathbb{S}_b\Big(\tfrac{Q}{2} + iA_2 + \tau\Big) F_\lambda\, \mathbb{S}_b^{-1}(Q - iC_2 + ip_u + \tau)\, \mathbb{S}_b^{-1}(Q - iC_2 - ip_u + \tau)\, F_\delta^T\, \mathbb{S}_b\Big(\tfrac{Q}{2} + iB_2 + \tau\Big),
\end{aligned} \tag{4.46}
$$

where:

$$F_\rho \;=\; \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\frac{i\pi}{4}} \end{pmatrix} \mathbf{1}_\rho \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\frac{i\pi}{4}} \end{pmatrix}^{-1} \;\Rightarrow\; F_{\mathrm{e}} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_{\mathrm{o}} \;=\; \begin{pmatrix} 0 & \mathrm{e}^{\frac{i\pi}{4}} \\ \mathrm{e}^{-\frac{i\pi}{4}} & 0 \end{pmatrix}.$$

Let us further denote:

$$\begin{aligned} \Phi^{(i)}_{\left[\substack{\mathrm{N\,N}\\ \mathrm{N\,N}}\right]}(p,\tau) &= \frac{S_{\mathrm{NS}}(Q/2+iA_i+\tau)S_{\mathrm{NS}}(Q/2+iB_i+\tau)}{S_{\mathrm{NS}}(Q-iC_i+ip+\tau)S_{\mathrm{NS}}(Q-iC_i-ip+\tau)}, \\ \Phi^{(i)}_{\left[\substack{\mathrm{R\,N}\\ \mathrm{N\,N}}\right]}(p,\tau) &= \frac{S_{\mathrm{R}}(Q/2+iA_i+\tau)S_{\mathrm{NS}}(Q/2+iB_i+\tau)}{S_{\mathrm{NS}}(Q-iC_i+ip+\tau)S_{\mathrm{NS}}(Q-iC_i-ip+\tau)}, \end{aligned}$$

e.t.c. and define:

$$\begin{aligned} \mathbf{\Phi}^{\mathrm{e}}_{(1)}(p,\tau) &= \begin{pmatrix} \Phi^{(1)}_{\left[\substack{\mathrm{N\,N}\\ \mathrm{N\,N}}\right]} & \mathrm{e}^{-\frac{i\pi}{2}}\Phi^{(1)}_{\left[\substack{\mathrm{N\,N}\\ \mathrm{R\,R}}\right]} \\ \Phi^{(1)}_{\left[\substack{\mathrm{R\,R}\\ \mathrm{R\,R}}\right]} & \mathrm{e}^{\frac{i\pi}{2}}\Phi^{(1)}_{\left[\substack{\mathrm{R\,R}\\ \mathrm{N\,N}}\right]} \end{pmatrix}(p,\tau), \quad \mathbf{\Phi}^{\mathrm{e}}_{(2)}(p,\tau) = \begin{pmatrix} \Phi^{(2)}_{\left[\substack{\mathrm{N\,N}\\ \mathrm{N\,N}}\right]} & \mathrm{e}^{\frac{i\pi}{2}}\Phi^{(2)}_{\left[\substack{\mathrm{N\,N}\\ \mathrm{R\,R}}\right]} \\ \Phi^{(2)}_{\left[\substack{\mathrm{R\,R}\\ \mathrm{R\,R}}\right]} & \mathrm{e}^{-\frac{i\pi}{2}}\Phi^{(2)}_{\left[\substack{\mathrm{R\,R}\\ \mathrm{N\,N}}\right]} \end{pmatrix}(p,\tau), \\ \mathbf{\Phi}^{\mathrm{o}}_{(1)}(p,\tau) &= \begin{pmatrix} \mathrm{e}^{\frac{i\pi}{4}}\Phi^{(1)}_{\left[\substack{\mathrm{N\,R}\\ \mathrm{N\,N}}\right]} & \mathrm{e}^{-\frac{i\pi}{4}}\Phi^{(1)}_{\left[\substack{\mathrm{N\,R}\\ \mathrm{R\,R}}\right]} \\ \mathrm{e}^{-\frac{i\pi}{4}}\Phi^{(1)}_{\left[\substack{\mathrm{R\,N}\\ \mathrm{R\,R}}\right]} & \mathrm{e}^{\frac{i\pi}{4}}\Phi^{(1)}_{\left[\substack{\mathrm{R\,N}\\ \mathrm{N\,N}}\right]} \end{pmatrix}(p,\tau), \; \mathbf{\Phi}^{\mathrm{o}}_{(2)}(p,\tau) = \begin{pmatrix} \mathrm{e}^{-\frac{i\pi}{4}}\Phi^{(2)}_{\left[\substack{\mathrm{N\,R}\\ \mathrm{N\,N}}\right]} & \mathrm{e}^{\frac{i\pi}{4}}\Phi^{(2)}_{\left[\substack{\mathrm{N\,R}\\ \mathrm{R\,R}}\right]} \\ \mathrm{e}^{\frac{i\pi}{4}}\Phi^{(2)}_{\left[\substack{\mathrm{R\,N}\\ \mathrm{R\,R}}\right]} & \mathrm{e}^{-\frac{i\pi}{4}}\Phi^{(2)}_{\left[\substack{\mathrm{R\,N}\\ \mathrm{N\,N}}\right]} \end{pmatrix}(p,\tau). \end{aligned} \tag{4.47}$$

It follows from (4.46) that (4.37) hold provided the relations

$$\mathrm{e}^{i\chi}\mathbf{\Phi}^{\gamma}_{(1)}(p_s,-\tau) = \int d\mu(t_2)\; \mathrm{e}^{-\frac{i\pi}{2}(bt_2)^2+\pi p_1 bt_2}\; \mathbf{\Phi}^{\gamma}_{(2)}(p_u,\tau)\cdot\left[\mathbf{B}^{\gamma,+}\right]^{\mathrm{T}} \tag{4.48}$$

are satisfied, where

$$i\chi = \frac{i\pi b^2}{2}(s_1^2-s^2)+\pi p_1 bs_2 - i\pi\alpha_1(\alpha_2+2bs_2),$$

and we have denoted (see (4.36))

$$\mathbf{B}^{\gamma,+} \;=\; \begin{pmatrix} \left[B^{\gamma,+}\right]^{\mathrm{e}}_{\mathrm{e}} & \left[B^{\gamma,+}\right]^{\mathrm{e}}_{\mathrm{o}} \\ \left[B^{\gamma,+}\right]^{\mathrm{o}}_{\mathrm{e}} & \left[B^{\gamma,+}\right]^{\mathrm{o}}_{\mathrm{o}} \end{pmatrix}.$$

Notice that we have traded (4.35) — a relation between unitary (for the parameters satisfying (4.41)) operators — for a relation between meromorphic functions. At this point we can analytically continue (4.48) to a "physical" values of the parameters,

$$\alpha_k \in \frac{Q}{2}+i\mathbb{R}, \qquad bs_k, bt_k \in -\frac{Q}{2}+i\mathbb{R}, \qquad k=1,2. \tag{4.49}$$

For α_k, s_k and t_k satisfying (4.49) all the parameters defined in (4.45) are real. Since for $x\in\mathbb{R}$:

$$|S_{\mathrm{NS}}\,(Q/2+ix)| \;=\; |S_{\mathrm{R}}\,(Q/2+ix)| \;=\; 1,$$

we get for $A_i, B_i, C_i \in \mathbb{R},\ \tau \in i\mathbb{R}$:

$$\Big(\boldsymbol{\Phi}^{\rm e}_{(1)}(p_u,\tau)\Big)^\dagger \cdot \boldsymbol{\Phi}^{\rm e}_{(1)}(p'_u,\tau) = \Big(\boldsymbol{\Phi}^{\rm o}_{(2)}(p_u,\tau)\Big)^\dagger \cdot \boldsymbol{\Phi}^{\rm o}_{(2)}(p'_u,\tau) \stackrel{\rm def}{=} \boldsymbol{\Theta}(p_u,p'_u;\tau).$$

Explicitly:

$$\begin{aligned}
\boldsymbol{\Theta}(p_u,p'_u;\tau)^{\rm e}{}_{\rm e} \;&=\; \boldsymbol{\Theta}(p_u,p'_u;\tau)^{\rm o}{}_{\rm o} \\
&= \left(\frac{S_{\rm NS}(ip_u-(\tau-iC_2))}{S_{\rm NS}(Q+ip_u+(\tau-iC_2))}\right)^\dagger \left(\frac{S_{\rm NS}(ip'_u-(\tau-iC_2))}{S_{\rm NS}(Q+ip'_u+(\tau-iC_2))}\right) \\
&+ \left(\frac{S_{\rm R}(ip_u-(\tau-iC_2))}{S_{\rm R}(Q+ip_u+(\tau-iC_2))}\right)^\dagger \left(\frac{S_{\rm R}(ip'_u-(\tau-iC_2))}{S_{\rm R}(Q+ip'_u+(\tau-iC_2))}\right) \\
&\equiv \left\langle ip_u \left|{}^{\rm N}_{\rm N}\right| \tau-iC_2\right\rangle \left\langle \tau-iC_2 \left|{}^{\rm N}_{\rm N}\right| ip'_u\right\rangle + \left\langle ip_u \left|{}^{\rm R}_{\rm R}\right| \tau-iC_2\right\rangle \left\langle \tau-iC_2 \left|{}^{\rm R}_{\rm R}\right| ip'_u\right\rangle ,
\end{aligned}$$

where in the last line we have borrowed the notation from [20], section 5.2. In the same notation:

$$\begin{aligned}
{\rm e}^{-\frac{i\pi}{2}}\boldsymbol{\Theta}(p_u,p'_u;\tau)^{\rm e}{}_{\rm o} \;&=\; {\rm e}^{\frac{i\pi}{2}}\boldsymbol{\Theta}(p_u,p'_u;\tau)^{\rm o}{}_{\rm e} \\
&= \left\langle ip_u \left|{}^{\rm N}_{\rm N}\right| \tau-iC_2\right\rangle \left\langle \tau-iC_2 \left|{}^{\rm R}_{\rm R}\right| ip'_u\right\rangle - \left\langle ip_u \left|{}^{\rm R}_{\rm R}\right| \tau-iC_2\right\rangle \left\langle \tau-iC_2 \left|{}^{\rm N}_{\rm N}\right| ip'_u\right\rangle
\end{aligned}$$

and, using [20] (equations (5.12) and (5.13)) we arrive at the orthogonality relation

$$\int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\;\boldsymbol{\Theta}(p_u,p'_u;\tau) \;=\; \frac{1}{\sinh\pi b p_u\,\sinh\pi b^{-1}p_u}\,\delta(p_u-p'_u)\,\mathbf{1}_{\rm e}. \tag{4.50}$$

For fixed p_1, α_i and s_i we see from (4.45) that the only parameter which changes with t_2 is p_u. In view of (4.50) it is therefore convenient to take $d\mu(t_2) = dp_u$. We thus get from (4.48):

$$\int\limits_{-i\infty}^{i\infty}\frac{d\tau}{i}\left(\boldsymbol{\Phi}^{\gamma}_{(2)}(p_u,\tau)\right)^\dagger \cdot \boldsymbol{\Phi}^{\gamma}_{(1)}(p_s,-\tau) = \frac{{\rm e}^{-\frac{i\pi}{2}(bt_2)^2+\pi p_1 bt_2 - i\chi}}{\sinh\pi b p_u\,\sinh\pi b^{-1}p_u}\left[\mathbf{B}^{\gamma,+}\right]^{\rm T} \tag{4.51}$$

with $bt_2 = i(p_u - p_1) - \alpha_2$.

There is a point concerning (4.51) which requires some care. Since $S_{\rm NS}(x)$ vanishes at $x = Q$ the functions

$$\Phi^{(i)}_{[\,:\,:\,]}(Q+iy+\tau), \qquad y\in\mathbb{R},$$

with $S_{\rm NS}(Q+iy+\tau)$ in the denominator have poles at the imaginary τ axis. Relations (4.50) and (4.51) are thus not well defined unless we specify the way the integration contour avoids these poles. To do this recall that the "physical" values of the parameters (4.49) were obtained in a process of analytic continuation from the purely imaginary values assumed in (4.41). It is immediate to see that if (4.41) holds, then the discussed poles are located

to the right from the imaginary axis. During the analytic continuation process no pole is allowed to cross the integration contour, so we take the contour in (4.50) and (4.51) to the left from the poles coming from the $S_{\rm NS}^{-1}$ factors. This coincides with the assumption made in [20] to derive the relation (4.50).

To present the un-normalized braiding matrices in the final form let us introduce one more set of (the most commonly used) variables:

$$a_1 = \frac{Q}{2} + ip_1,\;\; a_2 = \alpha_1,\;\; a_3 = \alpha_2,\;\; a_s = \frac{Q}{2} + ip_s,\;\; a_u = \frac{Q}{2} + ip_u,\;\; a_4 = a_3 + a_2 + a_1 + bs. \tag{4.52}$$

In these variables:

$$\begin{aligned} \frac{Q}{2} + iA_1 &= a_4 - a_3 + a_2, \quad & \frac{Q}{2} - iA_2 &= a_4 + a_3 - a_2, \\ \frac{Q}{2} + iB_1 &= \frac{Q}{2} + iB_2 & &= Q - a_1, \\ Q - iC_1 + ip_s &= a_s + a_2, \quad & Q - iC_1 - ip_s &= Q - a_s + a_2, \\ Q - iC_2 + ip_u &= a_u + a_3, \quad & Q - iC_2 - ip_u &= Q - a_u + a_3 \end{aligned}$$

and

$$\frac{i\pi}{2}(bt_2)^2 - \pi p_1 bt_2 + i\chi = i\pi\Big(\Delta_4 + \Delta_1 - \Delta_u - \Delta_s + a_3(a_4 - a_u) - a_2(a_4 - a_s)\Big) \stackrel{\rm def}{=} i\varphi_{a_s a_u}(a_3, a_2),$$

with $\Delta_i \equiv \Delta(a_i) = \frac{1}{2}a_i(Q - a_i)$. Using the relation

$$\sinh(\pi b p_u)\sinh(\pi b^{-1} p_u) \;=\; \frac{1}{4} S_{\rm NS}(2a_u) S_{\rm NS}(2Q - 2a_u)$$

and noticing that $\left[\mathbf{B}^{\gamma,+}\right]$ written as a function of $a_k,\; k = 1,2,3,4,s,u$ is nothing but the unnormalized braiding matrix we get from (4.51):

$$^{\mathbf{g}}\mathsf{B}^{\gamma,+}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right] = \mathrm{e}^{i\varphi_{a_s a_u}(a_3,a_2)} \frac{S_{\rm NS}(2a_u) S_{\rm NS}(2Q - 2a_u)}{4} \int\limits_{-i\infty}^{i\infty} \frac{d\tau}{i}\; \mathbf{J}^{\gamma}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right](\tau) \tag{4.53}$$

with

$$\mathbf{J}^{\gamma}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right](\tau) = \Big(\mathbf{\Phi}^{\gamma}_{(1)}(p_s, -\tau)\Big)^{\rm T} \cdot \overline{\mathbf{\Phi}^{\gamma}_{(2)}(p_u, \tau)}$$

and where, as in (4.36), we used the notation

$$^{\mathbf{g}}\mathsf{B}^{\rm e,+}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\rho}_{\;\eta} = {}^{\mathbf{g}}\mathsf{B}^{+}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\rho\rho}_{\;\;\eta\eta}, \qquad {}^{\mathbf{g}}\mathsf{B}^{\rm o,+}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\rho}_{\;\eta} = {}^{\mathbf{g}}\mathsf{B}^{+}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\rho\bar\rho}_{\;\;\eta\bar\eta}. \tag{4.54}$$

Explicitly:

$$\mathbf{J}^{\rm e}(\tau) = \begin{pmatrix} \left[\begin{smallmatrix} \rm N\,N\,N\,N \\ \rm N\,N\,N\,N \end{smallmatrix}\right](\tau) + \left[\begin{smallmatrix} \rm R\,R\,R\,R \\ \rm R\,R\,R\,R \end{smallmatrix}\right](\tau) & \frac{1}{i}\left[\begin{smallmatrix} \rm N\,N\,N\,N \\ \rm R\,R\,N\,N \end{smallmatrix}\right](\tau) + i\left[\begin{smallmatrix} \rm R\,R\,R\,R \\ \rm N\,N\,R\,R \end{smallmatrix}\right](\tau) \\ \frac{1}{i}\left[\begin{smallmatrix} \rm N\,N\,N\,N \\ \rm N\,N\,R\,R \end{smallmatrix}\right](\tau) + i\left[\begin{smallmatrix} \rm R\,R\,R\,R \\ \rm R\,R\,N\,N \end{smallmatrix}\right](\tau) & -\left[\begin{smallmatrix} \rm N\,N\,N\,N \\ \rm R\,R\,R\,R \end{smallmatrix}\right](\tau) - \left[\begin{smallmatrix} \rm R\,R\,R\,R \\ \rm N\,N\,N\,N \end{smallmatrix}\right](\tau) \end{pmatrix}$$

and

$$\mathbf{J}^{\rm o}(\tau) = \begin{pmatrix} i \begin{bmatrix} \mathrm{N\,R\,N\,R} \\ \mathrm{N\,N\,N\,N} \end{bmatrix}(\tau) + \frac{1}{i} \begin{bmatrix} \mathrm{R\,N\,R\,N} \\ \mathrm{R\,R\,R\,R} \end{bmatrix}(\tau) & \begin{bmatrix} \mathrm{N\,R\,N\,R} \\ \mathrm{R\,R\,N\,N} \end{bmatrix}(\tau) + \begin{bmatrix} \mathrm{R\,N\,R\,N} \\ \mathrm{N\,N\,R\,R} \end{bmatrix}(\tau) \\ \begin{bmatrix} \mathrm{N\,R\,N\,R} \\ \mathrm{N\,N\,R\,R} \end{bmatrix}(\tau) + \begin{bmatrix} \mathrm{R\,N\,R\,N} \\ \mathrm{R\,R\,N\,N} \end{bmatrix}(\tau) & \frac{1}{i} \begin{bmatrix} \mathrm{N\,R\,N\,R} \\ \mathrm{R\,R\,R\,R} \end{bmatrix}(\tau) + i \begin{bmatrix} \mathrm{R\,N\,R\,N} \\ \mathrm{N\,N\,N\,N} \end{bmatrix}(\tau) \end{pmatrix},$$

where the abbreviations

$$\begin{bmatrix} \mathrm{N\,N\,N\,N} \\ \mathrm{N\,N\,N\,N} \end{bmatrix}(\tau) = \frac{S_{\rm NS}(\bar{a}_4 - a_3 + a_2 + \tau) S_{\rm NS}(a_1 + \tau) S_{\rm NS}(a_4 - a_3 + a_2 + \tau) S_{\rm NS}(\bar{a}_1 + \tau)}{S_{\rm NS}(\bar{a}_u + \bar{a}_3 + \tau) S_{\rm NS}(a_u + \bar{a}_3 + \tau) S_{\rm NS}(a_s + a_2 + \tau) S_{\rm NS}(\bar{a}_s + a_2 + \tau)},$$

$$\begin{bmatrix} \mathrm{N\,R\,N\,R} \\ \mathrm{N\,N\,N\,N} \end{bmatrix}(\tau) = \frac{S_{\rm NS}(\bar{a}_4 - a_3 + a_2 + \tau) S_{\rm R}(a_1 + \tau) S_{\rm NS}(a_4 - a_3 + a_2 + \tau) S_{\rm R}(\bar{a}_1 + \tau)}{S_{\rm NS}(\bar{a}_u + \bar{a}_3 + \tau) S_{\rm NS}(a_u + \bar{a}_3 + \tau) S_{\rm NS}(a_s + a_2 + \tau) S_{\rm NS}(\bar{a}_s + a_2 + \tau)},$$

e.t.c. with $\bar{a}_i = Q - a_i$ have been applied.

Using the results of [20] it is straightforward to check that the functions $\mathbf{\Phi}^{\sigma}_{(i)}(p, \tau)$ satisfy a completeness relation of the form

$$\int_0^\infty dp_s \; \frac{S_{\rm NS}(2a_s) S_{\rm NS}(2Q - 2a_s)}{4} \; \left(\mathbf{\Phi}^{\gamma}_{(i)}(p_s, \tau)\right)^\dagger \cdot \mathbf{\Phi}^{\gamma}_{(i)}(p_s, \lambda) = \delta(\tau - \lambda)\, \mathbf{1}_{\rm e}. \tag{4.55}$$

From (4.53), (4.50) and (4.55) it then follows that

$$\int_0^\infty dp_s \; {\rm e}^{-2\pi i(\Delta_4 + \Delta_1 - \Delta_t - \Delta_s)}\, {}^{\mathbf{g}}\mathsf{B}^{\gamma,+}_{a_t a_s} \begin{bmatrix} a_2 & a_3 \\ a_4 & a_1 \end{bmatrix} \cdot {}^{\mathbf{g}}\mathsf{B}^{\gamma,+}_{a_s a_u} \begin{bmatrix} a_3 & a_2 \\ a_4 & a_1 \end{bmatrix} = \delta(p_t - p_u)\, \mathbf{1}_{\rm e}.$$

This equality implies that

$${\rm e}^{-2\pi i(\Delta_4 + \Delta_1 - \Delta_t - \Delta_s)}\, {}^{\mathbf{g}}\mathsf{B}^{\gamma,+}_{a_t a_s} \begin{bmatrix} a_2 & a_3 \\ a_4 & a_1 \end{bmatrix} = {}^{\mathbf{g}}\mathsf{B}^{\gamma,-}_{a_t a_s} \begin{bmatrix} a_2 & a_3 \\ a_4 & a_1 \end{bmatrix}$$

and allows to write

$${}^{\mathbf{g}}\mathsf{B}^{\gamma,\epsilon}_{a_s a_u} \begin{bmatrix} a_3 & a_2 \\ a_4 & a_1 \end{bmatrix} = {\rm e}^{i\pi\epsilon(\Delta_4 + \Delta_1 - \Delta_s - \Delta_u)}\; {}^{\mathbf{g}}\mathsf{B}^{\gamma}_{a_s a_u} \begin{bmatrix} a_3 & a_2 \\ a_4 & a_1 \end{bmatrix} \tag{4.56}$$

with $\epsilon = {\rm sign}(\sigma_2 - \sigma_1)$ and

$${}^{\mathbf{g}}\mathsf{B}^{\gamma}_{a_s a_u} \begin{bmatrix} a_3 & a_2 \\ a_4 & a_1 \end{bmatrix} = {\rm e}^{i\pi(a_3(a_4 - a_u) - a_2(a_4 - a_s))}\; \frac{S_{\rm NS}(2a_u) S_{\rm NS}(2Q - 2a_u)}{4} \int_{-i\infty}^{i\infty} \frac{d\tau}{i}\; \mathbf{J}^{\gamma}_{a_s a_u} \begin{bmatrix} a_3 & a_2 \\ a_4 & a_1 \end{bmatrix}(\tau).$$

To derive the other three braiding matrices which appear in (3.11) we use the realization of the vertex operator ${}^{\mathbf{g}}V_{a_3 a_1}(*\nu_\alpha|w)$ provided by the chiral descendants

$$*g^{\alpha,{\rm e}}_s(\sigma) = \{S_{-1/2}, g^{\alpha,{\rm o}}_s(\sigma)\} = -i\alpha\psi(\sigma) g^{\alpha,{\rm o}}_s(\sigma)$$

and

$$*g^{\alpha,{\rm o}}_s(\sigma) = [S_{-1/2}, g^{\alpha,{\rm e}}_s(\sigma)] = -i\alpha\psi(\sigma) g^{\alpha,{\rm e}}_s(\sigma).$$

Keeping in mind the definitions (4.52) and (4.45) we have on the one hand

$$g_{s_2}^{\alpha_2,\rho}(\sigma_2) *g_{s_1}^{\alpha_1,\eta}(\sigma_1) \;=\; \int\limits_{\frac{Q}{2}+i\mathbb{R}} \frac{d\alpha_u}{2i} \sum_{\lambda\delta} {}^{\mathbf{g}}\mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & *a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\rho\eta}{}_{\lambda\delta} *g_{t_1}^{\alpha_1,\lambda}(\sigma_1) g_{t_2}^{\alpha_2,\delta}(\sigma_2),$$

and on the other

$$\begin{aligned}
& g_{s_2}^{\alpha_2,\rho}(\sigma_2) *g_{s_1}^{\alpha_1,\eta}(\sigma_1) \\
&\quad = -i\alpha_1 g_{s_2}^{\alpha_2,\rho}(\sigma_2)\, \psi(\sigma_1)\, g_{s_1}^{\alpha_1,\bar{\eta}}(\sigma_1) \\
&\quad = (-1)^{|\rho|}(-i\alpha_1\psi(\sigma_1))\; g_{s_2}^{\alpha_2,\rho}(\sigma_2)\, g_{s_1}^{\alpha_1,\bar{\eta}}(\sigma_1) \\
&\quad = (-1)^{|\rho|} \int\limits_{\frac{Q}{2}+i\mathbb{R}} \frac{d\alpha_u}{2i} \sum_{\lambda,\delta} {}^{\mathbf{g}}\mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\rho\bar{\eta}}{}_{\lambda\delta}\; (-i\alpha_1\psi(\sigma_1)) g_{t_1}^{\alpha_1,\lambda}(\sigma_1) g_{t_2}^{\alpha_2,\delta}(\sigma_2) \\
&\quad = (-1)^{|\rho|} \int\limits_{\frac{Q}{2}+i\mathbb{R}} \frac{d\alpha_u}{2i} \sum_{\lambda\delta} {}^{\mathbf{g}}\mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\rho\bar{\eta}}{}_{\lambda\delta}\; *g_{t_1}^{\alpha_1,\bar{\lambda}}(\sigma_1) g_{t_2}^{\alpha_2,\delta}(\sigma_2) \\
&\quad = (-1)^{|\rho|} \int\limits_{\frac{Q}{2}+i\mathbb{R}} \frac{d\alpha_u}{2i} \sum_{\lambda\delta} {}^{\mathbf{g}}\mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\rho\bar{\eta}}{}_{\bar{\lambda}\delta}\; *g_{t_1}^{\alpha_1,\lambda}(\sigma_1) g_{t_2}^{\alpha_2,\delta}(\sigma_2).
\end{aligned}$$

Together with (4.56) this yields

$$ {}^{\mathbf{g}}\mathsf{B}_{a_s a_u}\left[\begin{smallmatrix} a_3 & *a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\rho\eta}{}_{\lambda\delta} \;=\; (-1)^{|\rho|}\; {}^{\mathbf{g}}\mathsf{B}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\rho\bar{\eta}}{}_{\bar{\lambda}\delta}\,. \tag{4.57}$$

Similar calculation also gives

$$ {}^{\mathbf{g}}\mathsf{B}_{a_s a_u}\left[\begin{smallmatrix} *a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\rho\eta}{}_{\lambda\delta} \;=\; (-1)^{|\eta|}\; {}^{\mathbf{g}}\mathsf{B}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\bar{\rho}\eta}{}_{\lambda\bar{\delta}} \tag{4.58}$$

and

$$ {}^{\mathbf{g}}\mathsf{B}_{a_s a_u}\left[\begin{smallmatrix} *a_3 & *a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\rho\eta}{}_{\lambda\delta} \;=\; (-1)^{|\rho|+|\eta|+1}\; {}^{\mathbf{g}}\mathsf{B}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\bar{\rho}\bar{\eta}}{}_{\bar{\lambda}\bar{\delta}}\,. \tag{4.59}$$

4.4 The normalized braiding matrices

Equations (4.33) and (4.34) can be presented in the form

$$\begin{aligned} N^{\eta}_{\mathbf{g}}(a_3,a_2,a_1) \;=\; & 2^{|\eta|}\left[\frac{1}{2}\Gamma\left(\frac{bQ}{2}\right) b^{-\frac{bQ}{2}}\right]^{\frac{a_3-a_1-a_2}{b}} \mathrm{e}^{\frac{i\pi}{2}(a_3-a_2-a_1)(Q-a_3+a_2-a_1)} \\ & \times\; \frac{\Gamma_\eta(a_3+a_1-a_2)\Gamma_\eta(\bar{a}_3+a_1-a_2)\Gamma_\eta(a_3+\bar{a}_1-a_2)\Gamma_\eta(\bar{a}_3+\bar{a}_1-a_2)}{\Gamma_{\rm NS}(Q)\Gamma_{\rm NS}(2a_1)\Gamma_{\rm NS}(Q-2a_2)\Gamma_{\rm NS}(2Q-2a_3)} \end{aligned} \tag{4.60}$$

where for $\eta = \mathrm{e}$ ($\eta = \mathrm{o}$) on the l.h.s. we have $\eta = \mathrm{NS}$ (resp. $\eta = \mathrm{R}$) on the r.h.s. From (3.6) we get:

$$\begin{aligned}
\mathsf{B}^{\mathrm{e},\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta}{}_{\rho} &= \frac{N^{\rho}_{\mathbf{g}}(a_4,a_2,a_u)N^{\rho}_{\mathbf{g}}(a_u,a_3,a_1)}{N^{\eta}_{\mathbf{g}}(a_4,a_3,a_s)N^{\eta}_{\mathbf{g}}(a_s,a_2,a_1)}\; {}^{\mathbf{g}}\mathsf{B}^{\mathrm{e},\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta}{}_{\rho}, \\
\mathsf{B}^{\mathrm{o},\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta}{}_{\rho} &= \frac{N^{\rho}_{\mathbf{g}}(a_4,a_2,a_u)N^{\bar{\rho}}_{\mathbf{g}}(a_u,a_3,a_1)}{N^{\eta}_{\mathbf{g}}(a_4,a_3,a_s)N^{\bar{\eta}}_{\mathbf{g}}(a_s,a_2,a_1)}\; {}^{\mathbf{g}}\mathsf{B}^{\mathrm{o},\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta}{}_{\rho},
\end{aligned}$$

so that using results from the previous subsection together with (4.60) we arrive at the explicit expression for the braiding matrix of the normalized chiral vertex operators defined in Eq. (3.11):

$$
\begin{aligned}
\mathsf{B}^{\gamma,\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right] &= \mathrm{e}^{\epsilon i\pi(\Delta_4+\Delta_1-\Delta_s-\Delta_u)}\, \mathsf{B}^{\gamma}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right], \qquad \gamma = \mathrm{e},\mathrm{o}, \quad \epsilon = \mathrm{sign}(\mathrm{Arg}\, z_3 - \mathrm{Arg}\, z_2), \\
\mathsf{B}^{\gamma}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right] &= \mathsf{M}^{\gamma}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right] \cdot \frac{\Gamma_{\mathrm{NS}}(2a_s)\Gamma_{\mathrm{NS}}(2Q-2a_s)}{4\Gamma_{\mathrm{NS}}(Q-2a_u)\Gamma_{\mathrm{NS}}(2a_u-Q)} \int_{-i\infty}^{i\infty} \frac{d\tau}{i}\, \mathbf{J}^{\gamma}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right](\tau), \qquad (4.61) \\
\mathsf{M}^{\mathrm{e}}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta}{}_{\rho} &= \frac{4^{|\rho|}\,\Gamma_\rho(a_u+a_4-a_2)\Gamma_\rho(\bar a_u+a_4-a_2)\Gamma_\rho(a_u+\bar a_4-a_2)\Gamma_\rho(\bar a_u+\bar a_4-a_2)}{4^{|\eta|}\,\Gamma_\eta(a_s+a_4-a_3)\Gamma_\eta(\bar a_s+a_4-a_3)\Gamma_\eta(a_s+\bar a_4-a_3)\Gamma_\eta(\bar a_s+\bar a_4-a_3)} \\
&\quad\times \frac{\Gamma_\rho(a_u+a_1-a_3)\Gamma_\rho(\bar a_u+a_1-a_3)\Gamma_\rho(a_u+\bar a_1-a_3)\Gamma_\rho(\bar a_u+\bar a_1-a_3)}{\Gamma_\eta(a_s+a_1-a_2)\Gamma_\eta(\bar a_s+a_1-a_2)\Gamma_\eta(a_s+\bar a_1-a_2)\Gamma_\eta(\bar a_s+\bar a_1-a_2)}, \\
\mathsf{M}^{\mathrm{o}}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta}{}_{\rho} &= \frac{\Gamma_\rho(a_u+a_4-a_2)\Gamma_\rho(\bar a_u+a_4-a_2)\Gamma_\rho(a_u+\bar a_4-a_2)\Gamma_\rho(\bar a_u+\bar a_4-a_2)}{\Gamma_\eta(a_s+a_4-a_3)\Gamma_\eta(\bar a_s+a_4-a_3)\Gamma_\eta(a_s+\bar a_4-a_3)\Gamma_\eta(\bar a_s+\bar a_4-a_3)} \\
&\quad\times \frac{\Gamma_{\bar\rho}(a_u+a_1-a_3)\Gamma_{\bar\rho}(\bar a_u+a_1-a_3)\Gamma_{\bar\rho}(a_u+\bar a_1-a_3)\Gamma_{\bar\rho}(\bar a_u+\bar a_1-a_3)}{\Gamma_{\bar\eta}(a_s+a_1-a_2)\Gamma_{\bar\eta}(\bar a_s+a_1-a_2)\Gamma_{\bar\eta}(a_s+\bar a_1-a_2)\Gamma_{\bar\eta}(\bar a_s+\bar a_1-a_2)}
\end{aligned}
$$

where, in analogy with (4.54),

$$
\mathsf{B}^{\mathrm{e},\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta}{}_{\rho} \equiv \mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta\eta}{}_{\rho\rho}, \qquad \mathsf{B}^{\mathrm{o},\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta}{}_{\rho} \equiv \mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta\bar\eta}{}_{\rho\bar\rho}.
$$

The form of the other braiding matrices can be worked out from (4.57) – (4.59) after taking into account (3.9). We get:

$$
\begin{aligned}
\mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & *a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta\rho}{}_{\lambda\delta} &= \frac{N^{\bar\lambda}_{\mathbf{g}}(a_4,a_2,a_u)N^{\delta}_{\mathbf{g}}(a_u,a_3,a_1)}{N^{\eta}_{\mathbf{g}}(a_4,a_3,a_s)N^{\bar\rho}_{\mathbf{g}}(a_s,a_2,a_1)}\, {}^{\mathbf{g}}\mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & *a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta\rho}{}_{\lambda\delta} \\
&= (-1)^{|\eta|}\, \frac{N^{\bar\lambda}_{\mathbf{g}}(a_4,a_2,a_u)N^{\delta}_{\mathbf{g}}(a_u,a_3,a_1)}{N^{\eta}_{\mathbf{g}}(a_4,a_3,a_s)N^{\bar\rho}_{\mathbf{g}}(a_s,a_2,a_1)}\, {}^{\mathbf{g}}\mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta\bar\rho}{}_{\bar\lambda\delta} \qquad (4.62) \\
&= (-1)^{|\eta|}\, \mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta\bar\rho}{}_{\bar\lambda\delta},
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} *a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta\rho}{}_{\lambda\delta} &= (-1)^{|\rho|}\, \mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\bar\eta\rho}{}_{\lambda\bar\delta}, \\
\mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} *a_3 & *a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\eta\rho}{}_{\lambda\delta} &= (-1)^{|\eta|+|\rho|+1}\mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\bar\eta\bar\rho}{}_{\bar\lambda\bar\delta}.
\end{aligned}
\qquad (4.63)
$$

4.5 Special braiding relations

In deriving the relations presented in Section 5 the limit $a_1 \to 0$ of the braiding matrix will be of a particular importance. To calculate it let us first of all note that $\lim_{a_1\to 0} V(a_1|z)$ is the identity operator so that

$$
\lim_{a_1\to 0} V^{\eta}_{a_s a_1}(\nu_2|z_2)V_{a_1 0}(\nu_1|z_1) = \delta^{\eta\mathrm{e}}\, V^{\mathrm{e}}_{a_2 0}(\nu_2|z_2).
$$

Consequently

$$\mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & 0 \end{smallmatrix}\right]^{\eta\mathrm{o}}{}_{\lambda\delta} = 0$$

and in the remaining cases the limit $\lim\limits_{a_1\to 0} \mathsf{B}^{\epsilon}_{a_s a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]$ is well defined only for $a_s = a_2$. In effect we only need to calculate

$$\lim_{a_1\to 0} \mathsf{B}^{\mathrm{e}}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\mathrm{e}}{}_{\lambda} \equiv \lim_{a_1\to 0} \mathsf{B}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\mathrm{ee}}{}_{\lambda\lambda}$$

and

$$\lim_{a_1\to 0} \mathsf{B}^{\mathrm{o}}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\mathrm{o}}{}_{\lambda} \equiv \lim_{a_1\to 0} \mathsf{B}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\mathrm{oe}}{}_{\lambda\bar{\lambda}}\,.$$

From (4.61) we have:

$$\begin{aligned}
&\mathsf{M}^{\mathrm{e}}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\mathrm{e}}{}_{\lambda} = \\
&\quad = 2^{|\lambda|}\frac{\Gamma_\lambda(a_u+a_4-a_2)\Gamma_\lambda(\bar{a}_u+a_4-a_2)\Gamma_\lambda(a_u+\bar{a}_4-a_2)\Gamma_\lambda(\bar{a}_u+\bar{a}_4-a_2)}{\Gamma_{\mathrm{NS}}(a_2+a_4-a_3)\Gamma_{\mathrm{NS}}(\bar{a}_2+a_4-a_3)\Gamma_{\mathrm{NS}}(a_2+\bar{a}_4-a_3)\Gamma_{\mathrm{NS}}(\bar{a}_2+\bar{a}_4-a_3)} \\
&\qquad \times \frac{\Gamma_\lambda(a_u+a_1-a_3)\Gamma_\lambda(\bar{a}_u+a_1-a_3)\Gamma_\lambda(a_u+\bar{a}_1-a_3)\Gamma_\lambda(\bar{a}_u+\bar{a}_1-a_3)}{\Gamma_{\mathrm{NS}}(a_1)\Gamma_{\mathrm{NS}}(Q+a_1-2a_2)\Gamma_{\mathrm{NS}}(\bar{a}_1)\Gamma_{\mathrm{NS}}(Q+\bar{a}_1-2a_2)}.
\end{aligned}$$

For $a_1 \to 0$ the factor $\Gamma_{\mathrm{NS}}(a_1)^{-1}$ present in this expression tends to zero and the braiding matrix is non-zero only for such values of the remaining parameters for which

$$\int_{-i\infty}^{i\infty} \frac{d\tau}{i}\; \mathbf{J}^{\mathrm{e}}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\mathrm{e}}{}_{\lambda} \tag{4.64}$$

provides a compensating, singular factor. Since components of $\mathbf{J}^{\mathrm{e}}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right](\tau)$ are meromorphic functions of τ (with the location of poles determined by the values of a_i) we can deform the integration contour in (4.64) such that it "keeps away" from the moving with $a_1 \to 0$ poles, arriving in the limit at a non-singular function of a_1, a_2, a_3, a_u and, in view of the discussion above, at a vanishing braiding matrix (cf. [9], Lemma 3). However, this procedure fails if the integration contour gets "pinched" between a pair of moving poles. In such a case we have to deform the contour past one of these colliding poles and the singular contribution can appear from the residue.

In our case the only pair of colliding poles appears in

$$\left[\begin{smallmatrix} \mathrm{N} & \mathrm{N} & \mathrm{N} & \mathrm{N} \\ \mathrm{N} & \mathrm{N} & \mathrm{N} & \mathrm{N} \end{smallmatrix}\right](\tau) = \frac{S_{\mathrm{NS}}(\bar{a}_4-a_3+a_2+\tau)S_{\mathrm{NS}}(a_1+\tau)S_{\mathrm{NS}}(a_4-a_3+a_2+\tau)S_{\mathrm{NS}}(\bar{a}_1+\tau)}{S_{\mathrm{NS}}(\bar{a}_u+\bar{a}_3+\tau)S_{\mathrm{NS}}(a_u+\bar{a}_3+\tau)S_{\mathrm{NS}}(2a_2+\tau)S_{\mathrm{NS}}(Q+\tau)},$$

a summand of $\mathbf{J}^{\mathrm{e}}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2 \\ a_4 & a_1 \end{smallmatrix}\right]^{\mathrm{e}}{}_{\mathrm{e}}$, where a pole at $\tau = -a_1$ (initially to the left of the contour) coming from a factor $S_{\mathrm{NS}}(a_1+\tau)$ approaches a pole of a function $S_{\mathrm{NS}}(Q+\tau)^{-1}$ at $\tau = 0^+$ (to the right of the contour). The residue at $\tau = -a_1$ gives a contribution to (4.64) of the form

$$I(a_1) = 2\frac{S_{\mathrm{NS}}(\bar{a}_4-a_3+a_2-a_1)S_{\mathrm{NS}}(a_4-a_3+a_2-a_1)S_{\mathrm{NS}}(Q-2a_1)}{S_{\mathrm{NS}}(\bar{a}_u+\bar{a}_3-a_1)S_{\mathrm{NS}}(a_u+\bar{a}_3-a_1)S_{\mathrm{NS}}(2a_2-a_1)S_{\mathrm{NS}}(Q-a_1)}$$

(in the course of calculating $I(a_1)$ the formula $\lim_{x\to 0} xS_{\rm NS}(x) = \pi^{-1}$ was used) and

$$\lim_{a_1\to 0} \mathsf{B}^{\rm e}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\rm e}{}_{\lambda} \;=\; \delta^{\rm e}_{\lambda}\frac{\Gamma_{\rm NS}(2a_2)\Gamma_{\rm NS}(2Q-2a_2)}{4\Gamma_{\rm NS}(Q-2a_u)\Gamma_{\rm NS}(2a_u-Q)}\lim_{a_1\to 0} I(a_1)\,\mathsf{M}^{\rm e}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\rm e}{}_{\rm e}$$

$$= \delta^{\rm e}_{\lambda}\,\frac{\Gamma_{\rm NS}(a_4+a_u-a_2)\Gamma_{\rm NS}(a_4+\bar a_u-a_2)\Gamma_{\rm NS}(\bar a_4+a_u-a_2)\Gamma_{\rm NS}(\bar a_4+\bar a_u-a_2)}{\Gamma_{\rm NS}(a_4+a_3-a_2)\Gamma_{\rm NS}(a_4+\bar a_3-a_2)\Gamma_{\rm NS}(\bar a_4+a_3-a_2)\Gamma_{\rm NS}(\bar a_4+\bar a_3-a_2)}$$
$$\times\,\frac{\Gamma_{\rm NS}(a_u-\bar a_3)\Gamma_{\rm NS}(\bar a_u-a_3)}{\Gamma_{\rm NS}(a_u-\bar a_u)\Gamma_{\rm NS}(\bar a_u-a_u)}\lim_{a_1\to 0}\frac{\Gamma_{\rm NS}(a_u-a_3+a_1)\Gamma_{\rm NS}(a_3-a_u+a_1)}{2\Gamma_{\rm NS}(Q)\Gamma_{\rm NS}(2a_1)}.$$

Since

$$a_u = \frac{Q}{2}+ip_u,\qquad a_3=\frac{Q}{2}+ip_3,\qquad \lim_{x\to 0} x\Gamma_{\rm NS}(x) = \frac{\Gamma_{\rm NS}(Q)}{\pi},$$

we have

$$\lim_{a_1\to 0}\frac{\Gamma_{\rm NS}(a_u-a_3+a_1)\Gamma_{\rm NS}(a_3-a_u+a_1)}{2\Gamma_{\rm NS}(Q)\Gamma_{\rm NS}(2a_1).} = \frac{1}{\pi}\lim_{a_1\to 0}\frac{a_1}{a_1^2+(p_u-p_3)^2} \;=\; \delta(p_u-p_3)$$

and

$$\lim_{a_1\to 0} \mathsf{B}^{\rm e}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\rm e}{}_{\lambda} \;=\; \delta^{\rm e}_{\lambda}\,\delta(p_u-p_3).$$

Calculations leading to

$$\lim_{a_1\to 0} \mathsf{B}^{\rm o}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\rm o}{}_{\lambda}$$

are analogous. This time the colliding pair of poles appears in the function

$$i\left[\begin{smallmatrix}\rm R & \rm N & \rm R & \rm N\\ \rm N & \rm N & \rm N & \rm N\end{smallmatrix}\right](\tau) \;=\; i\frac{S_{\rm R}(\bar a_4-a_3+a_2+\tau)S_{\rm NS}(a_1+\tau)S_{\rm R}(a_4-a_3+a_2+\tau)S_{\rm NS}(\bar a_1+\tau)}{S_{\rm NS}(\bar a_u+\bar a_3+\tau)S_{\rm NS}(a_u+\bar a_3+\tau)S_{\rm NS}(a_s+a_2+\tau)S_{\rm NS}(\bar a_s+a_2+\tau)},$$

being a summand of $\mathbf{J}^{\rm o}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\rm o}{}_{\rm o}$. Computing the residue at $\tau=-a_1$ and taking the limit $a_1\to 0$ we get:

$$\lim_{a_1\to 0} \mathsf{B}^{\rm o}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\rm o}{}_{\lambda} \;=\; i\delta^{\rm o}_{\lambda}\,\delta(p_u-p_3).$$

Summarizing:

$$\lim_{a_1\to 0} \mathsf{B}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\rho\eta}{}_{\lambda\delta} \;=\; i^{|\rho|}\,\delta^{\rho}_{\lambda}\delta^{\eta\rm e}\delta_{\delta\rm e}\,\delta(p_u-p_3) \tag{4.65}$$

or equivalently

$$V^{\rho}_{a_4 a_2}(\nu_3|z_3)V^{\eta}_{a_2 0}(\nu_2|z_2) \;=\; i^{|\rho|}\,\Omega^{\epsilon_{32}}_{432}\,\delta^{\eta\rm e}\,V^{\rho}_{a_4 a_3}(\nu_2|z_2)V^{\rm e}_{a_3 0}(\nu_3|z_3), \tag{4.66}$$

where

$$\Omega^{\epsilon_{32}}_{432} \;=\; {\rm e}^{i\pi\epsilon_{32}(\Delta_4-\Delta_3-\Delta_2)}.$$

Finally, the relations between "starred" and "un-starred" braiding matrices, Eqs. (4.62) and (4.63), together with (4.65) and (4.66) give:

$$\lim_{a_1\to 0} \mathsf{B}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & *a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\rho\eta}{}_{\lambda\delta} = (-1)^{|\rho|}\lim_{a_1\to 0} \mathsf{B}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\rho\bar\eta}{}_{\bar\lambda\delta} \;=\; (-i)^{|\rho|}\,\delta^{\rho}_{\bar\lambda}\delta^{\eta\rm o}\delta_{\delta\rm e}\,\delta(p_u-p_3),$$

$$\lim_{a_1\to 0} \mathsf{B}_{a_2 a_u}\left[\begin{smallmatrix} *a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\rho\eta}{}_{\lambda\delta} = (-1)^{|\eta|}\lim_{a_1\to 0} \mathsf{B}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\bar\rho\eta}{}_{\lambda\bar\delta} \;=\; i^{|\bar\rho|}\,\delta^{\rho}_{\bar\lambda}\delta^{\eta\rm e}\delta_{\delta\rm o}\,\delta(p_u-p_3),$$

$$\lim_{a_1\to 0} \mathsf{B}_{a_2 a_u}\left[\begin{smallmatrix} *a_3 & *a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\rho\eta}{}_{\lambda\delta} = (-1)^{|\bar\rho|+|\eta|}\lim_{a_1\to 0} \mathsf{B}_{a_2 a_u}\left[\begin{smallmatrix} a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\bar\rho\bar\eta}{}_{\bar\lambda\bar\delta} \;=\; -(-i)^{|\bar\rho|}\,\delta^{\rho}_{\lambda}\delta^{\eta\rm o}\delta_{\delta\rm o}\,\delta(p_u-p_3),$$

or equivalently:

$$
\begin{aligned}
V^{\rho}_{a_4a_2}(\nu_3|z_3)V^{\eta}_{a_20}(*\nu_2|z_2) &= (-i)^{|\rho|}\,\Omega^{\epsilon_{32}}_{432}\,\delta^{\eta\mathrm{o}}\,V^{\bar\rho}_{a_4a_3}(*\nu_2|z_2)V^{\mathrm{e}}_{a_30}(\nu_3|z_3),\\
V^{\rho}_{a_4a_2}(*\nu_3|z_3)V^{\eta}_{a_20}(\nu_2|z_2) &= i^{|\bar\rho|}\,\Omega^{\epsilon_{32}}_{432}\,\delta^{\eta\mathrm{e}}\,V^{\bar\rho}_{a_4a_3}(\nu_2|z_2)V^{\mathrm{o}}_{a_30}(*\nu_3|z_3),\\
V^{\rho}_{a_4a_2}(*\nu_3|z_3)V^{\eta}_{a_20}(*\nu_2|z_2) &= -(-i)^{|\bar\rho|}\,\Omega^{\epsilon_{32}}_{432}\,\delta^{\eta\mathrm{o}}\,V^{\rho}_{a_4a_3}(*\nu_2|z_3)V^{\mathrm{o}}_{a_30}(*\nu_3|z_3).
\end{aligned}
$$

5. Braiding and fusion properties of the Neveu-Schwarz blocks

We shall define, following [16], four even and four odd NS conformal blocks

$$
\mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix}\_a_3 & \_a_2\\ a_4 & a_1\end{smallmatrix}\right](z) \,=\, \langle\nu_4\big|V^{\eta}_{a_4a_s}(\_\nu_3|1)V^{\eta}_{a_sa_1}(\_\nu_2|z)\nu_1\rangle. \tag{5.1}
$$

This choice is motivated by the observation that the knowledge of (5.1) is sufficient (once the relevant three point coupling constants are know) to compute all the four-point correlation functions in the NS sector of a given $N=1$ SCFT. Inserting between the chiral vertices a projection operator onto the basis of $\mathcal{V}_s$ formed by the vectors $\nu_{s,IK}$ (see (3.3)) we get:

$$
\mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix}\_a_3 & \_a_2\\ a_4 & a_1\end{smallmatrix}\right](z) = \sum_{IK,JL}\rho\,{}^{a_4}_{\infty}{}^{a_3}_{1}{}^{a_s}_{0}(\nu_4,\_\nu_3,\nu_{s,IK})\,\big[G^{\eta}_{a_s}\big]^{IK,JL}\,\rho\,{}^{a_s}_{\infty}{}^{a_2}_{z}{}^{a_2}_{0}(\nu_{s,JL},\_\nu_2,\nu_1).
$$

Here the three-linear form ρ is defined by

$$
\rho\,{}^{a_3}_{\infty}{}^{a_2}_{z}{}^{a_1}_{0}(\xi_3,\xi_2,\xi_1) = \big\langle\xi_3\big|V_{a_3a_1}(\xi_2|z)\xi_1\big\rangle\,, \qquad \xi_i\in\mathcal{V}_i,\ \ i=1,2,3,
$$

$[G^{\eta}_{a_s}]^{IK,JL}$ is an element of the matrix inverse to

$$
\big[G^{\eta}_{a_s}\big]_{IK,JL} = \langle\nu_{s,IK}|\nu_{s,JL}\rangle
$$

and $(-1)^{|\eta|}=(-1)^{|K|}=(-1)^{|L|}$.

Notice that

$$
\begin{aligned}
&\big\langle\nu_4\big|V^{\eta}_{a_4a_s}(\_\nu_3|z_3)V^{\eta}_{a_sa_1}(\_\nu_2|z_2)V_{a_10}(\nu_1|z_1)\nu_0\big\rangle\\
&\quad= \big\langle\nu_4\big|V^{\eta}_{a_4a_s}(\_\nu_3|z_{31})V^{\eta}_{a_sa_1}(\_\nu_2|z_{21})\nu_1\big\rangle \qquad\qquad (5.2)\\
&\quad= z_{31}^{\Delta_4-\_\Delta_3-\_\Delta_2-\Delta_1}\,\big\langle\nu_4\big|V^{\eta}_{a_4a_s}(\_\nu_3|1)V^{\eta}_{a_sa_1}(\_\nu_2|z)\nu_1\big\rangle = z_{31}^{\Delta_4-\_\Delta_3-\_\Delta_2-\Delta_1}\,\mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix}\_a_3 & \_a_2\\ a_4 & a_1\end{smallmatrix}\right](z),
\end{aligned}
$$

where $z=\frac{z_{21}}{z_{31}}$ with $z_{ij}=z_i-z_j$, $*\Delta\equiv\Delta+\frac{1}{2}$ and $\_\Delta$ which may stand for Δ or $*\Delta$, and

$$
\begin{aligned}
&\big\langle\nu_4\big|V_{a_4a_1}\big(V^{\eta}_{a_ta_2}(\_\nu_3|z_{32})\nu_2|z_2\big)\,V_{a_10}(\_\nu_1|z_1)\nu_0\big\rangle\\
&\quad= \big\langle\nu_4\big|V_{a_4a_1}\big(V^{\eta}_{a_ta_2}(\_\nu_3|z_{32})\nu_2|z_{21}\big)\,\_\nu_1\big\rangle \qquad\qquad (5.3)\\
&\quad= \sum_{IK,JL}\rho\,{}^{a_4}_{\infty}{}^{a_t}_{z_{21}}{}^{a_1}_{0}(\nu_4,\nu_{t,IK},\_\nu_1)[G^{\eta}_{a_t}]^{IK,JL}\rho\,{}^{a_t}_{\infty}{}^{a_3}_{z_{32}}{}^{a_2}_{0}(\nu_{t,JL},\_\nu_3,\nu_2)\\
&\quad= \mathrm{e}^{-i\pi\varepsilon_{32}(\Delta_t-\_\Delta_3-\Delta_2+\frac{1}{2}|\eta|)}\,z_{21}^{\Delta_4-\_\Delta_3-\Delta_2-\_\Delta_1}\,\mathcal{F}^{\eta}_{a_t}\left[\begin{smallmatrix}\_a_1 & \_a_3\\ a_4 & a_2\end{smallmatrix}\right]\big(1-z^{-1}\big)\,,
\end{aligned}
$$

where $\epsilon_{32} = \mathrm{sgn}(\mathrm{Arg}\, z_{32})$. To derive (5.2) and (5.3) we used the state-operator correspondence

$$\xi = V_{a0}(\xi|0)\nu_0, \qquad \xi \in \mathcal{V}_a,$$

which follows from (3.5) and the definition of the generalized chiral vertex operator, the fact that L_{-1} acts (see (3.7)) as a generator of translation in z and consequently

$$V_{a_3a_1}(\xi_2|z) = \mathrm{e}^{zL_{-1}} V_{a_3a_1}(\xi_2|0)\mathrm{e}^{-zL_{-1}},$$

and the identities

$$\begin{aligned}
\rho^{a_3\;a_2\;a_1}_{\infty\;z\;0}(\xi_3,\xi_2,\xi_1) &= z^{\Delta(\xi_3)-\Delta(\xi_2)-\Delta(\xi_1)}\rho^{a_3\;a_2\;a_1}_{\infty\;1\;0}(\xi_3,\xi_2,\xi_1), && L_0\xi_i = \Delta(\xi_i)\xi_i,\\
\rho^{a_3\;a_2\;a_1}_{\infty\;z\;0}(\nu_3,\nu_{2,IK},_\nu_1) &= (-1)^{|I|+|K|}\,\rho^{a_3\;a_1\;a_t}_{\infty\;z\;0}(\nu_3,_\nu_1,\nu_{2,IK}), && |K| \in \mathbb{N},\\
\rho^{a_3\;a_2\;a_1}_{\infty\;z\;0}(\nu_3,\nu_{2,IK},_\nu_1) &= (-1)^{|I|+|K|-\frac{1}{2}}\,\rho^{a_3\;a_1\;a_2}_{\infty\;z\;0}(\nu_3,_\nu_1,\nu_{2,IK}), && |K| \in \mathbb{N}+\frac{1}{2},
\end{aligned}$$

which are also easily derived using definitions from the subsection 3.2 and the commutation relations (3.8).

With the help of relations (3.11) and (5.2) it is immediate to arrive at the $s-u$ braiding relations satisfied by the NS blocks. We have:

$$\begin{aligned}
\mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix}_a_3 & _a_2\\ a_4 & a_1\end{smallmatrix}\right](z) &= z_{31}^{-\Delta_4+_\Delta_3+_\Delta_2+\Delta_1}\left\langle \nu_4\middle|V^{\eta}_{a_4a_s}(_\nu_3|z_{31})V^{\eta}_{a_sa_1}(_\nu_2|z_{21})\nu_1\right\rangle\\
&= z_{31}^{-\Delta_4+_\Delta_3+_\Delta_2+\Delta_1}\int\limits_{\mathbb{S}}\frac{da_u}{2i}\sum_{\rho=\,\mathrm{e,o}}\mathsf{B}^{\epsilon}_{a_sa_u}\left[\begin{smallmatrix}_a_3 & _a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\eta\eta}{}_{\rho\rho}\left\langle \nu_4\middle|V^{\rho}_{a_4a_u}(_\nu_2|z_{21})V^{\rho}_{a_sa_u}(_\nu_3|z_{31})\nu_1\right\rangle\\
&= z^{\Delta_4-_\Delta_3-_\Delta_2-\Delta_1}\int\limits_{\mathbb{S}}\frac{da_u}{2i}\sum_{\rho=\,\mathrm{e,o}}\mathsf{B}^{\epsilon}_{a_sa_u}\left[\begin{smallmatrix}_a_3 & _a_2\\ a_4 & a_1\end{smallmatrix}\right]^{\eta\eta}{}_{\rho\rho}\,\mathcal{F}^{\rho}_{a_u}\left[\begin{smallmatrix}_a_2 & _a_3\\ a_4 & a_1\end{smallmatrix}\right]\left(z^{-1}\right).
\end{aligned}$$

Special braiding relations (4.66) allow to derive a generalization of the Euler's relation satisfied by the hypergeometric function. W have:

$$\begin{aligned}
\mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix}a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right](z) &= z_{31}^{\Delta_1+\Delta_2+\Delta_3-\Delta_4}\langle\nu_4|V^{\eta}_{a_4a_s}(\nu_3|z_3)V^{\eta}_{a_sa_1}(\nu_2|z_2)V^{\mathrm{e}}_{a_10}(\nu_1|z_1)\nu_0\rangle\\
&= i^{|\eta|}\,\mathrm{e}^{i\pi\epsilon_{21}(\Delta_s-\Delta_2-\Delta_1)}\,z_{31}^{\Delta_1+\Delta_2+\Delta_3-\Delta_4}\,\langle\nu_4|V^{\eta}_{a_4a_s}(\nu_3|z_3)V^{\eta}_{a_sa_2}(\nu_1|z_1)V^{\mathrm{e}}_{a_20}(\nu_2|z_2)\nu_0\rangle\\
&= i^{|\eta|}\,\mathrm{e}^{i\pi\epsilon_{21}(\Delta_s-\Delta_2-\Delta_1)}\,\left(\frac{z_{32}}{z_{31}}\right)^{\Delta_4-\Delta_3-\Delta_2-\Delta_1}\,\mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix}a_3 & a_1\\ a_4 & a_2\end{smallmatrix}\right]\left(\frac{z_{12}}{z_{32}}\right).
\end{aligned}$$

If we exclude the situation when $\arg z_1$ lies between $\arg z_2$ and $\arg z_3$ then

$$\epsilon_{21} \;=\; -\mathrm{sign}(\arg z) \;\equiv -\epsilon$$

and we get

$$\mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix}a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right](z) \;=\; i^{|\eta|}\,\mathrm{e}^{-i\pi\epsilon(\Delta_s-\Delta_2-\Delta_1)}\,(1-z)^{\Delta_4-\Delta_3-\Delta_2-\Delta_1}\,\mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix}a_3 & a_1\\ a_4 & a_2\end{smallmatrix}\right]\left(\tfrac{z}{z-1}\right). \tag{5.4}$$

Similarly

$$\mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix} a_3 & *a_2 \\ a_4 & a_1 \end{smallmatrix}\right](z) = z_{31}^{\Delta_1 + *\Delta_2 + \Delta_3 - \Delta_4} \langle \nu_4 | V^{\eta}_{a_4 a_s}(\nu_3|z_3) V^{\eta}_{a_s a_1}(*\nu_2|z_2) V^{\mathrm{e}}_{a_1 0}(\nu_1|z_1)\nu_0\rangle$$
$$= i^{|\bar\eta|}\, \mathrm{e}^{i\pi\epsilon_{21}(\Delta_s - \Delta_2 - \Delta_1)}\, z_{31}^{\Delta_1 + *\Delta_2 + \Delta_3 - \Delta_4}\, \langle \nu_4 | V^{\eta}_{a_4 a_s}(\nu_3|z_3) V^{\bar\eta}_{a_s a_2}(\nu_1|z_1) V^{\mathrm{o}}_{a_2 0}(*\nu_2|z_2)\nu_0\rangle$$
$$= i^{|\bar\eta|}\, \mathrm{e}^{i\pi\epsilon_{21}(\Delta_s - \Delta_2 - \Delta_1)} \left(\frac{z_{32}}{z_{31}}\right)^{\Delta_4 - \Delta_3 - *\Delta_2 - \Delta_1} \mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix} a_3 & a_1 \\ a_4 & *a_2 \end{smallmatrix}\right]\left(\frac{z_{12}}{z_{32}}\right)$$

where

$$\mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix} a_3 & a_1 \\ a_4 & *a_2 \end{smallmatrix}\right](x) \equiv \langle \nu_4 | V^{\eta}_{a_4 a_s}(\nu_3|1) V^{\bar\eta}_{a_s a_2}(\nu_1|x) S_{-\frac{1}{2}}\nu_2\rangle = \mathcal{F}^{\bar\eta}_{a_s}\left[\begin{smallmatrix} *a_3 & a_1 \\ a_4 & a_2 \end{smallmatrix}\right](x) + (-1)^{|\eta|}\mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix} a_3 & *a_1 \\ a_4 & a_2 \end{smallmatrix}\right](x),$$

and with the same restriction on the arguments of z_i, $i = 1, 2, 3$:

$$\mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix} a_3 & *a_2 \\ a_4 & a_1 \end{smallmatrix}\right](z) \;=\; i^{|\bar\eta|}\, \mathrm{e}^{-i\pi\epsilon(\Delta_s - \Delta_2 - \Delta_1)}\, (1-z)^{\Delta_4 - \Delta_3 - *\Delta_2 - \Delta_1}\, \mathcal{F}^{\eta}_{a_s}\left[\begin{smallmatrix} a_3 & a_1 \\ a_4 & *a_2 \end{smallmatrix}\right]\left(\tfrac{z}{z-1}\right) \tag{5.5}$$

It shouldn't be difficult for the reader to derive the Euler's relations for the remaining four blocks.

Denote graphically the identity

$$V_{a_3 a_1}(\nu_2|z_2) V_{a_1 0}(\nu_1|z_1)\nu_0 = \mathrm{e}^{z_1 L_{-1}} V_{a_3 a_1}(\nu_2|z_{21})\nu_1 = V_{a_3 0}\big(V_{a_3 a_1}(\nu_2|z_{21})\nu_1|z_1\big)\nu_0,$$

which expresses the operator-state correspondence for the chiral vertex operators, as

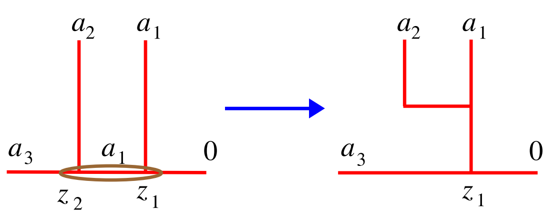

Fig. 2 Graphical notation for the elementary fusion transformation

and consider now the following sequence of "moves":

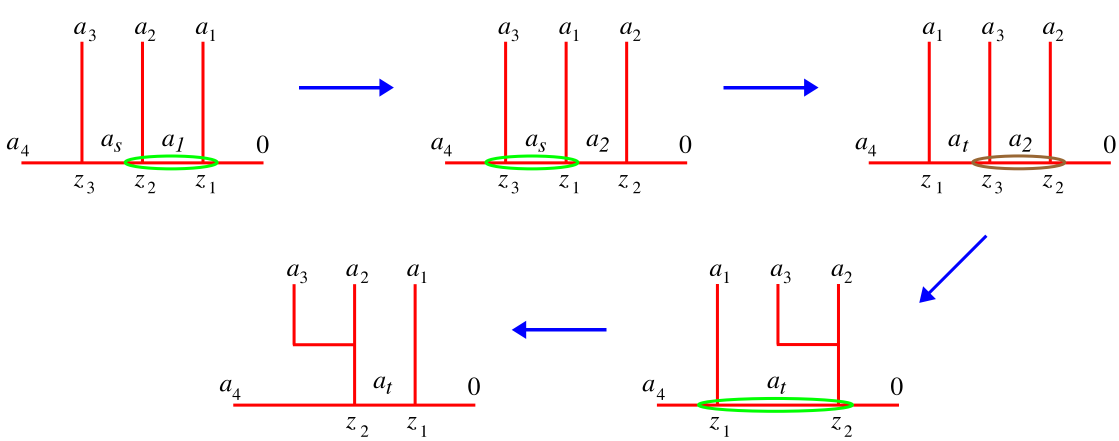

Fig. 3 Moves which lead to $s-t$ braiding

It results in an identity:

$$
\begin{aligned}
&\langle\nu_4|V^\eta_{a_4a_s}(\nu_3|z_3)V^\eta_{a_sa_1}(\nu_2|z_2)V^{\rm e}_{a_10}(\nu_1|z_1)\nu_0\rangle\\
&\quad= i^{|\eta|}\Omega^{\epsilon_{21}}_{s21}\langle\nu_4|V^\eta_{a_4a_s}(\nu_3|z_3)V^\eta_{a_sa_2}(\nu_1|z_1)V^{\rm e}_{a_20}(\nu_2|z_2)\nu_0\rangle\\
&\quad= i^{|\eta|}\Omega^{\epsilon_{21}}_{s21}\int\limits_{\frac{Q}{2}+i\mathbb{R}}\frac{da_t}{2i}\;\mathsf{B}^{\epsilon_{31}}_{a_sa_t}\!\left[\begin{smallmatrix}a_3 & a_1\\ a_4 & a_2\end{smallmatrix}\right]^{\eta\eta}_{\;\;\rho\rho}\langle\nu_4|V^\rho_{a_4a_t}(\nu_1|z_1)V^\rho_{a_ta_2}(\nu_3|z_3)V^{\rm e}_{a_20}(\nu_2|z_2)\nu_0\rangle\\
&\quad= i^{|\eta|}\Omega^{\epsilon_{21}}_{s21}\int\limits_{\frac{Q}{2}+i\mathbb{R}}\frac{da_t}{2i}\;\mathsf{B}^{\epsilon_{31}}_{a_sa_t}\!\left[\begin{smallmatrix}a_3 & a_1\\ a_4 & a_2\end{smallmatrix}\right]^{\eta\eta}_{\;\;\rho\rho}\langle\nu_4|V^\rho_{a_4a_t}(\nu_1|z_1)V^\rho_{a_20}\big(V^\rho_{a_ta_2}(\nu_3|z_{32})\nu_2|z_2\big)\,\nu_0\rangle\\
&\quad= i^{|\eta|}\Omega^{\epsilon_{21}}_{s21}\int\limits_{\frac{Q}{2}+i\mathbb{R}}\frac{da_t}{2i}\;\mathsf{B}^{\epsilon_{31}}_{a_sa_t}\!\left[\begin{smallmatrix}a_3 & a_1\\ a_4 & a_2\end{smallmatrix}\right]^{\eta\eta}_{\;\;\rho\rho}(-i)^{|\rho|}\Omega^{\epsilon_{12}}_{4t1}\,\langle\nu_4|V^{\rm e}_{a_4a_1}\big(V^\rho_{a_ta_2}(\nu_3|z_{32})\nu_2|z_2\big)\,V^{\rm e}_{a_10}(\nu_1|z_1)\nu_0\rangle
\end{aligned}
\tag{5.6}
$$

and with the same restriction on the arguments of z_i, $i=1,2,3$, as above:

$$
\Omega^{\epsilon_{21}}_{s21}\,\Omega^{\epsilon_{12}}_{4t2}\,\mathsf{B}^{\epsilon_{31}}_{a_sa_t}\!\left[\begin{smallmatrix}a_3 & a_1\\ a_4 & a_2\end{smallmatrix}\right]^{\eta\eta}_{\;\;\rho\rho}={\rm e}^{i\pi(\epsilon_{31}-\epsilon_{21})(\Delta_4+\Delta_2-\Delta_s-\Delta_t)}\,\mathsf{B}_{a_sa_t}\!\left[\begin{smallmatrix}a_3 & a_1\\ a_4 & a_2\end{smallmatrix}\right]^{\eta\eta}_{\;\;\rho\rho}=\mathsf{B}_{a_sa_t}\!\left[\begin{smallmatrix}a_3 & a_1\\ a_4 & a_2\end{smallmatrix}\right]^{\eta\eta}_{\;\;\rho\rho}.
\tag{5.7}
$$

Using (5.3) and (5.4) we get

$$
\langle\nu_4|V^{\rm e}_{a_4a_1}\big(V^\rho_{a_ta_2}(\nu_3|z_{32})\nu_2|z_2\big)\,V^{\rm e}_{a_10}(\nu_1|z_1)\nu_0\rangle={\rm e}^{\frac{i\pi}{2}(1-\epsilon_{32})|\rho|}\,z_{31}^{\Delta_4-\Delta_3-\Delta_2-\Delta_1}\,\mathcal{F}^\rho_{a_t}\left[\begin{smallmatrix}a_1 & a_2\\ a_4 & a_3\end{smallmatrix}\right](1-z)
$$

so that if we define (as in Section 2) the fusion matrix F through the relation

$$
\mathcal{F}^\eta_{a_s}\left[\begin{smallmatrix}-a_3 & -a_2\\ -a_4 & -a_1\end{smallmatrix}\right](z)=\int\limits_{\frac{Q}{2}+i\mathbb{R}}\frac{da_t}{2i}\;\mathsf{F}_{a_sa_t}\left[\begin{smallmatrix}-a_3 & -a_2\\ -a_4 & -a_1\end{smallmatrix}\right]^\eta_{\;\;\rho}\mathcal{F}^\rho_{a_t}\left[\begin{smallmatrix}-a_1 & -a_2\\ -a_4 & -a_3\end{smallmatrix}\right](1-z)
\tag{5.8}
$$

then (5.2), (5.6) and (5.7) yield

$$
\mathsf{F}_{a_sa_t}\left[\begin{smallmatrix}a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]^\eta_{\;\;\rho}={\rm e}^{\frac{i\pi}{2}(|\eta|-\epsilon_{32}|\rho|)}\,\mathsf{B}_{a_sa_t}\!\left[\begin{smallmatrix}a_3 & a_1\\ a_4 & a_2\end{smallmatrix}\right]^{\eta\eta}_{\;\;\rho\rho}.
\tag{5.9}
$$

If we choose in the complex z_2 plane a cut from z_3 to $+\infty$, then

$$
\epsilon_{32}={\rm sign}(\arg z_3-\arg z_2)<0
$$

and the formula for the fusion matrix acquires its final form

$$
\mathsf{F}_{a_sa_t}\left[\begin{smallmatrix}a_3 & a_2\\ a_4 & a_1\end{smallmatrix}\right]^\eta_{\;\;\rho}={\rm e}^{\frac{i\pi}{2}(|\eta|+|\rho|)}\,\mathsf{B}_{a_sa_t}\!\left[\begin{smallmatrix}a_3 & a_1\\ a_4 & a_2\end{smallmatrix}\right]^{\eta\eta}_{\;\;\rho\rho}.
\tag{5.10}
$$

Similarly, application of the sequence of moves (5.6) to the correlator

$$
\langle\nu_4|V^\eta_{a_4a_s}(\nu_3|z_3)V^\eta_{a_sa_1}(*\nu_2|z_2)V^{\rm e}_{a_10}(\nu_1|z_1)\nu_0\rangle
$$

followed by the use of the Euler's formula (5.5) gives:

$$
\mathsf{F}_{a_sa_t}\left[\begin{smallmatrix}a_3 & *a_2\\ a_4 & a_1\end{smallmatrix}\right]^\eta_{\;\;\rho}={\rm e}^{-\frac{i\pi}{2}(|\eta|+|\bar\rho|)}\,\mathsf{B}_{a_sa_t}\!\left[\begin{smallmatrix}a_3 & a_1\\ a_4 & a_2\end{smallmatrix}\right]^{\eta\bar\eta}_{\;\;\rho\bar\rho}.
\tag{5.11}
$$

Note that the form of the fusion matrix (5.10) with the braiding matrix given by (4.61) coincides[6] with the one conjectured in [20]. This, in view of the properties of this matrix proven in [20] which include:

- its invariance under the substitution $a_i \to Q - a_i,\ i = 1, 2, 3, 4, s, t$ (which is equivalent to the statement that the fusion matrix depends on a_i only through conformal weights $\Delta(a_i)$) and under exchange of its "rows" (i.e. $(a_3, a_2) \leftrightarrow (a_4, a_1)$) or "columns" (i.e. $(a_3, a_4) \leftrightarrow (a_2, a_1)$);
- calculation of its values at the limit $a_2 \to -b$ and
- the fact, that the matrix (5.10) satisfies the orthogonality relation (2.7),

provides a rather strong argument for the consistency of the SLFT.

With the result above it is straightforward to derive the form of the fusion matrices for the remaining blocks (5.1) and to check (using the orthogonality and completeness relations for the functions $\mathbf{\Phi}$ defined in Eq. (4.47)) that they indeed possess the properties which ensure the validity of the bootstrap equations for the remaining four-point NS correlators.

6. Conclusions and prospects

Results from the quantum Liouville field theory have a number of applications, to name only the continuous approach to the two dimensional quantum gravity, where the choice of conformal gauge for the two-dimensional metric leads to the theory of coupled Liouville and matter fields ([27] or, more recently, [28]), quantization of Teichmüller space of Riemann surfaces [29] and a relation between Liouville field theory and the H_3^+ WZNW model [30–34] which resulted among others in a proof of the crossing symmetry of the latter theory. More specifically, the fusion matrix of conformal blocks was shown [35, 36] to be related to the three point correlation function of the boundary operators in the Liouville theory, what allowed for a full solution of the boundary SLFT.

Some of these results were demonstrated to possess counterparts in the supersymmetric case. The supersymmetric Liouville gravity was discussed already in [37] (see also [38] for some new results on the subject); in [39] a link in the spirit of [34] between the supersymmetric Liouville field theory and the WZNW models on the OSP$(p|2)$, $p = 1, 2$, supergroups was constructed and used to derive an explicit formulas for the two- and three-point functions in the OSP$(1|2)$ WZNW model. Once the fusion matrix of the NS blocks is explicitly known, it seems not to be difficult to calculate the (so far unknown) three point function of the boundary operators in the NS sector of the supersymmetric Liouville theory and check a crossing symmetry in a (sector of) the OSP$(1|2)$ WZNW model as well.

[6] One needs to take into account a factor $4^{|\eta|-|\rho|}$ which comes from a different normalizations of conformal blocks and a transposition which follows from a different definitions of a fusion matrix adapted, cf. Eq. (5.8) from the present paper and Eq. (2.6) from [20].

In this work we did not touch upon the second, Ramond sector of the SLFT. Due to the square root singularity of the correlation functions of the Ramond fields and the tensor $S(z)$ the analysis of the conformal blocks in the Ramond sector is considerably more difficult than in the NS sector and the four-point conformal blocks with the external Ramond states seem to have been defined and discussed only recently [40]. In spite of this, it seems not to be difficult to modify the constructions of the present work to include the Ramond sector as well, completing in this way a proof of the consistency of the $N = 1$ supersymmetric Liouville field theory [41].

Acknowledgments

The work of L.H. is partially supported by MNII grant 189/6.PRUE/2007/7 and by the Polish State Research Committee (KBN) grant no. N N202 0859 33.

D.C. is grateful to the faculty of the Institute of Physics, Jagiellonian University, Kraków, for the hospitality.

A. Special functions related to the Barnes double gamma function

For $\Re x > 0$ the function $\Gamma_b(x)$ has an integral representation of the form:

$$\log \Gamma_b(x) \;=\; \int\limits_0^\infty \frac{dt}{t}\left[\frac{\mathrm{e}^{-xt}-\mathrm{e}^{-\frac{Q}{2}t}}{\left(1-\mathrm{e}^{-tb}\right)\left(1-\mathrm{e}^{-t/b}\right)} - \frac{\left(\frac{Q}{2}-x\right)^2}{2\mathrm{e}^t} - \frac{\frac{Q}{2}-x}{t}\right].$$

$\Gamma_b(x)$ is (up to the normalizing factor $\Gamma^{-1}_{b,b^{-1}}(Q/2)$) a special case of the Barnes double gamma function $\Gamma_{\omega_1,\omega_2}(x)$ with $\omega_1 = \omega_2^{-1} = b$, being an analytic continuation of the function

$$\log \Gamma_{\omega_1,\omega_2}(x;s) = \frac{\partial}{\partial s}\sum_{m,n=0}^{\infty}(m\omega_1 + n\omega_2 + z)^{-s}$$

to $s = 0$. Both expressions explicitly show an important self-duality property,

$$\Gamma_b(x) = \Gamma_{b^{-1}}(x).$$

$\Gamma_b(x)$ satisfies functional relations

$$\Gamma_b(x+b) = \frac{\sqrt{2\pi}\, b^{bx-\frac{1}{2}}}{\Gamma(bx)}\Gamma_b(x), \qquad \Gamma_b\left(x+b^{-1}\right) = \frac{\sqrt{2\pi}\, b^{-\frac{x}{b}+\frac{1}{2}}}{\Gamma(\frac{x}{b})}\Gamma_b(x), \tag{A.1}$$

and can be analytically continued to the whole complex x plane as a meromorphic function with no zeroes and with poles located at $x = -mb - n\frac{1}{b},\; m, n \in \mathbb{N}$. Relations (A.1) allow to calculate residues of these poles in terms of $\Gamma_b(Q)$; for instance for $x \to 0$:

$$\Gamma_b(x) \;=\; \frac{\Gamma_b(Q)}{2\pi x} + \mathcal{O}(1).$$

It is convenient to introduce

$$\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}, \qquad S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}, \qquad G_b(x) = \mathrm{e}^{-\frac{i\pi}{2}x(Q-x)} S_b(x), \quad \text{(A.2)}$$

and, borrowing the notation from [26], to denote:

$$\begin{aligned} \Gamma_{\mathrm{NS}}(x) &= \Gamma_b\left(\frac{x}{2}\right)\Gamma_b\left(\frac{x+Q}{2}\right), \qquad \Gamma_{\mathrm{R}}(x) = \Gamma_b\left(\frac{x+b}{2}\right)\Gamma_b\left(\frac{x+b^{-1}}{2}\right), \\ \Upsilon_{\mathrm{NS}}(x) &= \Upsilon_b\left(\frac{x}{2}\right)\Upsilon_b\left(\frac{x+Q}{2}\right), \qquad \Upsilon_{\mathrm{R}}(x) = \Upsilon_b\left(\frac{x+b}{2}\right)\Upsilon_b\left(\frac{x+b^{-1}}{2}\right), \end{aligned} \quad \text{(A.3)}$$

etc.

Using relations (A.1) and definitions (A.2), (A.3) one can easily establish basic properties of these functions. In the paper we used:

- Relations between S and G functions:

$$G_{\mathrm{NS}}(x) = \zeta_0\, \mathrm{e}^{-\frac{i\pi}{4}x(Q-x)} S_{\mathrm{NS}}(x), \qquad G_{\mathrm{R}}(x) = \mathrm{e}^{-\frac{i\pi}{4}}\zeta_0\, \mathrm{e}^{-\frac{i\pi}{4}x(Q-x)} S_{\mathrm{R}}(x), \qquad \text{(A.4)}$$

where $\zeta_0 = \mathrm{e}^{-\frac{i\pi Q^2}{8}}$.

- Shift relations:

$$G_{\mathrm{NS}}(x+b^{\pm 1}) = \left(1+\mathrm{e}^{i\pi b^{\pm 1}x}\right) G_{\mathrm{R}}(x), \qquad G_{\mathrm{R}}(x+b^{\pm 1}) = \left(1-\mathrm{e}^{i\pi b^{\pm 1}x}\right) G_{\mathrm{NS}}(x). \quad \text{(A.5)}$$

- Reflection properties:

$$S_{\mathrm{NS}}(x) S_{\mathrm{NS}}(Q-x) = S_{\mathrm{R}}(x) S_{\mathrm{R}}(Q-x) = 1$$

- Locations of zeroes and poles:

$$\begin{aligned} S_{\mathrm{NS}}(x) &= 0 \quad \Leftrightarrow \quad x = Q + mb + nb^{-1}, \quad m,n \in \mathbb{Z}_{\geq 0},\ m+n \in 2\mathbb{Z}, \\ S_{\mathrm{R}}(x) &= 0 \quad \Leftrightarrow \quad x = Q + mb + nb^{-1}, \quad m,n \in \mathbb{Z}_{\geq 0},\ m+n \in 2\mathbb{Z}+1, \\ S_{\mathrm{NS}}(x)^{-1} &= 0 \quad \Leftrightarrow \quad x = -mb - nb^{-1}, \quad m,n \in \mathbb{Z}_{\geq 0},\ m+n \in 2\mathbb{Z}, \\ S_{\mathrm{R}}(x)^{-1} &= 0 \quad \Leftrightarrow \quad x = -mb - nb^{-1}, \quad m,n \in \mathbb{Z}_{\geq 0},\ m+n \in 2\mathbb{Z}+1. \end{aligned}$$

- Basic residue:

$$\lim_{x \to 0} x\, S_{\mathrm{NS}}(x) = \frac{1}{\pi}. \qquad \text{(A.6)}$$

B. Weyl-type representation of the screening charges

Positive operators $(\mathsf{Q}^c_{\rm I})^2$ and $(\mathsf{Q}_{\rm I})^2$ can be represented in a form

$$(\mathsf{Q}^c_{\rm I})^2 \;=\; {\rm e}^{2b\mathsf{u}}, \qquad (\mathsf{Q}_{\rm I})^2 \;=\; {\rm e}^{-2\pi b\mathsf{v}}$$

where u and v are hermitian on $\mathcal{H} = \mathcal{H}_{\mathsf{B}} \otimes \mathcal{F}_{\rm F}$ with the standard scalar product. From the braiding relation

$$(\mathsf{Q}^c_{\rm I})^2\,(\mathsf{Q}_{\rm I})^2 \;=\; {\rm e}^{-4i\pi b^2}\,(\mathsf{Q}_{\rm I})^2\,(\mathsf{Q}^c_{\rm I})^2\,, \tag{B.1}$$

we get

$$[\mathsf{u},\mathsf{v}] \;=\; i. \tag{B.2}$$

Let us now define hermitian operators x and s such that

$$\begin{aligned} \mathsf{u} &= \mathsf{x} - \frac{\pi}{2}\,\mathsf{s}, \\ \pi\mathsf{v} &= -\mathsf{x} - \frac{\pi}{2}\,\mathsf{s} + \pi\mathsf{p} \end{aligned}$$

and additionally $[\mathsf{s},\mathsf{x}] = [\mathsf{s},\mathsf{p}] = 0$. Taking into account (B.2) we get

$$[\mathsf{x},\mathsf{p}] = i$$

and

$$\begin{aligned} (\mathsf{Q}^c_{\rm I})^2 &= {\rm e}^{2b\mathsf{u}} \;=\; {\rm e}^{-\pi b\mathsf{s}}{\rm e}^{2b\mathsf{x}} \;=\; \left({\rm e}^{-\frac{1}{2}\pi b\mathsf{s}}{\rm e}^{b\mathsf{x}}\right)^2, \\ (\mathsf{Q}_{\rm I})^2 &= {\rm e}^{-2\pi b\mathsf{v}} \;=\; {\rm e}^{\pi b\mathsf{s}}{\rm e}^{2b\mathsf{x}-2\pi b\mathsf{p}} \;=\; \left({\rm e}^{\frac{1}{2}\pi b\mathsf{s}}{\rm e}^{\frac{1}{2}b\mathsf{x}}{\rm e}^{-\pi b\mathsf{p}}{\rm e}^{\frac{1}{2}b\mathsf{x}}\right)^2. \end{aligned}$$

so that

$$\mathsf{Q}^c_{\rm I} = \eta^c\,{\rm e}^{-\frac{1}{2}\pi b\mathsf{s}}{\rm e}^{b\mathsf{x}}, \qquad \mathsf{Q}_{\rm I} \;=\; \eta\,{\rm e}^{\frac{1}{2}\pi b\mathsf{s}}{\rm e}^{\frac{1}{2}b\mathsf{x}}{\rm e}^{-\pi b\mathsf{p}}{\rm e}^{\frac{1}{2}b\mathsf{x}}$$

where

$$[\eta,\mathsf{x}] \;=\; [\eta,\mathsf{s}] \;=\; [\eta,\mathsf{p}] \;=\; [\eta^c,\mathsf{x}] \;=\; [\eta^c,\mathsf{s}] \;=\; [\eta^c,\mathsf{p}] \;=\; 0$$

and

$$\{\eta,\eta^c\} = 0, \quad \eta^2 = (\eta^c)^2 = 1. \tag{B.3}$$

The Hilbert space $\mathcal{H}$ has a natural $\mathbb{Z}_2$ grading given by $(-1)^{\mathsf{F}}$, where

$$\mathsf{F} = \sum_{k\in\mathbb{N}+\frac{1}{2}} \psi_{-k}\psi_k$$

is the fermion number and $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ with $(-1)^{\mathsf{F}}|\psi^\pm\rangle = \pm|\psi^\pm\rangle$ for $|\psi^\pm\rangle \in \mathcal{H}^\pm$. The screening charges are odd with respect to this grading,

$$\mathsf{Q}_{\rm I},\, \mathsf{Q}^c_{\rm I} \;:\; \mathcal{H}^\pm \;\mapsto\; \mathcal{H}^\mp,$$

and we can represent them in a form

$$
\begin{aligned}
\mathsf{Q}_{\mathrm{I}}^{c}\begin{pmatrix}|\psi^+\rangle\\|\psi^-\rangle\end{pmatrix} &= \mathrm{e}^{-\frac{1}{2}\pi b\mathsf{s}}\mathrm{e}^{b\mathsf{x}}\begin{pmatrix}0 & \eta_{\mathrm{eo}}^{c}\\ \eta_{\mathrm{oe}}^{c} & 0\end{pmatrix}\begin{pmatrix}|\psi^+\rangle\\|\psi^-\rangle\end{pmatrix},\\
\mathsf{Q}_{\mathrm{I}}\begin{pmatrix}|\psi^+\rangle\\|\psi^-\rangle\end{pmatrix} &= \mathrm{e}^{\frac{1}{2}\pi b\mathsf{s}}\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\mathrm{e}^{-\pi b\mathsf{p}}\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\begin{pmatrix}0 & \eta_{\mathrm{eo}}\\ \eta_{\mathrm{oe}} & 0\end{pmatrix}\begin{pmatrix}|\psi^+\rangle\\|\psi^-\rangle\end{pmatrix}.
\end{aligned}
$$

The conditions (B.3) together with the hermiticity of Q−s thus gives

$$
\eta^c = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \eta = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}
$$

up to a similarity transformation. Replacing $\mathsf{s} \to i\mathsf{t}$ we get (4.43).

References

[1] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Infinite Conformal Symmetry In Two-Dimensional Quantum Field Theory*, Nucl. Phys. B **241**, 333 (1984).

[2] H. Dorn and H. J. Otto, *Two and three point functions in Liouville theory*, Nucl. Phys. B **429** (1994) 375 [arXiv:hep-th/9403141].

[3] A. B. Zamolodchikov and Al. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, Nucl. Phys. B **477**, 577 (1996) [arXiv:hep-th/9506136].

[4] Al. Zamolodchikov, *Two-dimensional conformal symmetry and critical four-spin correlation functions in the Ashkin-Teller model*, Sov. Phys. JETP **63** (1986) 1061.

[5] Al. Zamolodchikov, *Two-dimensional conformal symmetry and critical four-spin correlation functions in the Ashkin-Teller model*, Sov. Phys. JETP **63** (1986) 1061.

[6] Al. Zamolodchikov, *Conformal symmetry in two-dimensional space: recursion representation of conformal block*, Theor. Math. Phys. **73** (1987) 1088.

[7] G. W. Moore and N. Seiberg, *Polynomial Equations For Rational Conformal Field Theories*, Phys. Lett. B **212** (1988) 451;

G. W. Moore and N. Seiberg, *Classical And Quantum Conformal Field Theory*, Commun. Math. Phys. **123** (1989) 177.

[8] B. Ponsot and J. Teschner, *Liouville bootstrap via harmonic analysis on a noncompact quantum group*, arXiv:hep-th/9911110.

[9] B. Ponsot and J. Teschner, *Clebsch-Gordan and Racah-Wigner coefficients for a continuous series of representations of U_q(sl(2,R))*, Commun. Math. Phys. **224**, 613 (2001) [arXiv:math/0007097].

[10] J. Teschner, *Liouville theory revisited*, Class. Quant. Grav. **18**, R153 (2001) [arXiv:hep-th/0104158].

[11] J. Teschner, *A lecture on the Liouville vertex operators*, Int. J. Mod. Phys. A **19S2**, 436 (2004) [arXiv:hep-th/0303150].

[12] D. Friedan, Z. a. Qiu and S. H. Shenker, *Superconformal Invariance In Two-Dimensions And The Tricritical Ising Model*, Phys. Lett. B **151**, 37 (1985).

[13] M. A. Bershadsky, V. G. Knizhnik and M. G. Teitelman, *Superconformal Symmetry In Two-Dimensions*, Phys. Lett. B **151** (1985) 31.

[14] A. B. Zamolodchikov and R. G. Pogosian, *Operator algebra in two-dimensional superconformal field theory*, Sov. J. Nucl. Phys. **47** (1988) 929 [Yad. Fiz. **47** (1988) 1461].

[15] R. H. Poghosian, *Structure constants in the N = 1 super-Liouville field theory*, Nucl. Phys. B **496**, 451 (1997) [arXiv:hep-th/9607120];

R. C. Rashkov and M. Stanishkov, *Three-point correlation functions in* $N = 1$ *Super Liouville Theory*, Phys. Lett. B **380**, 49 (1996) [arXiv:hep-th/9602148].

[16] L. Hadasz, Z. Jaskolski and P. Suchanek, *Recursion representation of the Neveu-Schwarz superconformal block*, JHEP **0703**, 032 (2007) [arXiv:hep-th/0611266].

[17] V. A. Belavin, *N = 1 SUSY conformal block recursive relations*, arXiv:hep-th/0611295.

[18] A. Belavin, V. Belavin, A. Neveu and A. Zamolodchikov, *Bootstrap in supersymmetric Liouville field theory. I: NS sector*, Nucl. Phys. B **784**, 202 (2007) [arXiv:hep-th/0703084].

[19] V. A. Belavin, *On the N = 1 super Liouville four-point functions*, Nucl. Phys. B **798**, 423 (2008) [arXiv:0705.1983 [hep-th]].

[20] L. Hadasz, *On the fusion matrix of the N=1 Neveu-Schwarz blocks*, JHEP **0712**, 071 (2007) [arXiv:0707.3384 [hep-th]].

[21] Y. Nakayama, *Liouville field theory: A decade after the revolution,* Int. J. Mod. Phys. A **19** (2004) 2771 [arXiv:hep-th/0402009].

[22] E. Frenkel and D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*, Mathematical Surveys and Monographs, vol. 88, AMS, 2000.

[23] V. Kac, *Vertex algebra for beginners*, AMS,1998.

[24] G. Felder, J. Frohlich and G. Keller, *On the structure of unitary conformal field theory 1. Existence of conformal blocks*, Commun. Math. Phys. **124**, 417 (1989);

G. Felder, J. Frohlich and G. Keller, *Braid matrices and structure constants for minimal conformal models* Commun. Math. Phys. **124**, 647 (1989);

[25] V. S. Dotsenko and V. A. Fateev, *Conformal algebra and multipoint correlation functions in 2D statistical models*, Nucl. Phys. B **240**, 312 (1984);

V. S. Dotsenko and V. A. Fateev, *Four Point Correlation Functions And The Operator Algebra In The Two-Dimensional Conformal Invariant Theories With The Central Charge* $C < 1$, Nucl. Phys. B **251**, 691 (1985).

[26] T. Fukuda and K. Hosomichi, *Super Liouville theory with boundary*, Nucl. Phys. B **635**, 215 (2002) [arXiv:hep-th/0202032].

[27] A. M. Polyakov, *Quantum geometry of bosonic strings*, Phys. Lett. B **103** (1981) 207.

[28] A. A. Belavin and A. B. Zamolodchikov, *Integrals Over Moduli Spaces, Ground Ring, And Four-Point Function In Minimal Liouville Gravity*, Theor. Math. Phys. **147** (2006) 729.

[29] J. Teschner, *On the relation between quantum Liouville theory and the quantized Teichmüller spaces*, Int. J. Mod. Phys. A **19S2**, 459 (2004) [arXiv:hep-th/0303149].

[30] J. Teschner, *Crossing symmetry in the H(3)+ WZNW model*, Phys. Lett. B **521**, 127 (2001) [arXiv:hep-th/0108121].

[31] B. Ponsot, *Monodromy of solutions of the Knizhnik-Zamolodchikov equation: SL(2,C)/SU(2) WZNW model*, Nucl. Phys. B **642**, 114 (2002) [arXiv:hep-th/0204085].

[32] S. Ribault and J. Teschner, *H_3^+ WZNW correlators from Liouville theory*, JHEP **0506**, 014 (2005) [arXiv:hep-th/0502048].

[33] K. Hosomichi and S. Ribault, *Solution of the H_3^+ model on a disc*, JHEP **0701**, 057 (2007) [arXiv:hep-th/0610117].

[34] Y. Hikida and V. Schomerus, *H_3^+ WZNW model from Liouville field theory*, arXiv:0706.1030 [hep-th].

[35] J. Teschner, *Remarks on Liouville theory with boundary*, arXiv:hep-th/0009138.

[36] B. Ponsot and J. Teschner, *Boundary Liouville field theory: Boundary three point function*, Nucl. Phys. B **622**, 309 (2002) [arXiv:hep-th/0110244].

[37] A. M. Polyakov, *Quantum geometry of fermionic strings*, Phys. Lett. B **103** (1981) 211.

[38] A. Belavin and V. Belavin, *Four-point function in Super Liouville Gravity*, arXiv:0810.1023 [hep-th].

[39] Y. Hikida and V. Schomerus, *Structure constants of the $OSP(1|2)$ WZNW model*, JHEP **0712** (2007) 100 [arXiv:0711.0338 [hep-th]].

[40] L. Hadasz, Z. Jaskólski and P. Suchanek, *Elliptic recurrence representation of the N=1 superconformal blocks in the Ramond sector*, arXiv:0810.1203 [hep-th], to appear in the JHEP.

[41] L. Hadasz and D. Chorążkiewicz, work in progress.